BOUNDARY CONTROL FOR THE 2-D WAVE EQUATION ON CURVED POLYGONS

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▪ ABSTRACT: In this work we study exact boundary controllability for the 2-D wave equation in curved polygonal domains with control of Neumann type. The initial state is supposed to have finite energy and the control obtained is square integrable

▪ KEYWORDS: Exact boundary control; wave equation; Neumann boundary control; nonsmooth domain.

1 Introduction

Throughout the paper we will consider the usual Sobolev spaces H^s, H_0^s, H_{loc}^s, $s \geq 0$, $H^0 = L^2$,... etc, as in Dautray and Lions (1988). Let Ω be a bounded domain in R^n with boundary Γ. Let $\nu(x)$ be the unit vector normal to Γ, outward with respect to Ω, defined for almost every $x \in \Gamma$.

An interesting exact controllability problem for the wave equation is the following: given a domain $\Omega \subset R^n$ and a positive integer m, find $T > 0$ such that for every initial state $(u_0, u_1) \in H^m(\Omega) \times H^{m-1}(\Omega)$, there exists a control function g so that the solution $u \in H^m(\Omega \times [0,T])$ of the system

$$u_{tt} - \Delta u = 0 \qquad in\ \Omega \times [0,T] \tag{1}$$
$$u(x,0) = u_0(x) \qquad in\ \Omega \tag{2}$$
$$u_t(x,0) = u_1(x) \qquad in\ \Omega \tag{3}$$
$$\alpha u + \beta \frac{\partial u}{\partial \nu} = g \qquad on\ \Gamma \times [0,T] \tag{4}$$

satisfies the final condition

$$u(x,T) = \frac{\partial u}{\partial \nu}(x,T) = 0 \qquad in\ \Omega \tag{5}$$

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Here $x = (x_1, ...x_n)$, $u_t = \frac{\partial u}{\partial t}$, $u_{tt} = \frac{\partial^2 u}{\partial t^2}$, $\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$, $\frac{\partial u}{\partial \nu}$ is the normal derivative of u , α and β are real constants satisfying $\alpha^2 + \beta^2 \neq 0$.

The physical meaning of this problem is the following: suppose that a vibration determined by the initial state (u_0, u_1) is occurring inside the region Ω. The problem is to find $T > 0$ and a way to act on the boundary of the region in order to make the vibration cease completely by time T. The final condition (5) states that particle x, at time T, is in the equilibrium position $u = 0$ with null velocity. If $n = 2$, we can regard Ω as a vibrating membrane and g as a force acting on its contour Γ, to drive the membrane to rest after time T has elapsed.

In Russell (1973), assuming that $\Gamma \in C^\infty$ and $m \geq 2$, the exact controllability problem above was solved with control g in $H^{m-\frac{2}{3}}(\Gamma \times [0, T])$. Later (see Lions (1988)), assuming that Γ was at least $C^2$and that Ω was starshapped, the same problem was solved for $m = 1$ and control in $L^2(\Gamma \times [0, T])$. The case where Ω is a straight polygonal or polyhedral domain was considered in Grisvard (1989), but the control obtained there had to have Dirichlet action on a nonempty part of the boundary.

In the present paper we study the exact controllability problem above assuming that the domain Ω satisfies the following conditions:

(a) $\Omega \subset R^2$ is bounded and simply connected,
(b) Γ is a closed piecewise C^∞ curve with no self intersection,
(c) Ω lies entirely on one side of Γ,
(d) Γ has no cusps.

Such domain will be referred to as a *curved polygon*. Requirements (b),(c) and (d) make Ω a Lispchitz domain (particularly, it satisfies the uniform cone condition). This suffices to assure the well-posedness of the mixed problems considered throughout the paper. Observe that the class of domains considered here includes the straight polygons as well as the domains with C^∞ boundaries.

In the next few sections we will prove the following result: *given a curved polygon Ω there exists $T > diam(\Omega)$ so that, for every initial state $(u_0, u_1) \in H^1(\Omega) \times L^2(\Omega)$ there exists a control function $g \in L^2(\Gamma \times [0, T])$ such that the solution of the system (1)-(4) satisfies the final condition (5).* Our result generalizes the corresponding ones in Russell (1973), Lions (1988) and Grisvard (1989). The method employed here was introduced in Russell (1973). Two decisive steps are the local energy decay of the solution of the Cauchy problem for the homogeneous wave equation with initial data in $H^1(R^2) \times L^2(R^2)$ and the regularity of the traces of such solution along the surface $\Gamma \times [0, T]$. In section 2 we prove the energy decay and establish the traces as a corollary of a theorem due to D. Tataru (see Tataru (1998)). Section 3 is dedicated to the proof of the main result.

It is worth noticing that a similar result holds true for curved polyhedral domains $\Omega \subset R^3$. In this case, Huygens' principle makes the proof very easy and will not be included here.

2 Local Decay and Trace Regularity

Let φ, $\psi \in C_0^\infty(R^n)$ be two arbitrary functions and $T > 0$ an arbitrary real number. It is well known (see John (1986)) that the Cauchy problem

$$w_{tt} - \Delta w = 0 \qquad in\ R^n \times R \tag{6}$$
$$w(x,0) = \varphi(x) \qquad in\ R^n \tag{7}$$
$$w_t(x,0) = \psi(x) \qquad in\ R^n \tag{8}$$

has a unique solution $w \in C^\infty(R^{n+1})$ with support in the set

$$\{(x,t); \exists\ y \in\ supp\varphi \cup supp\psi,\ \ |x-y| \leq t\}.$$

The classical theory of the wave equation gives the following estimate

$$\|w(.,t)\|^2_{H^1(R^n)} + \|w_t(.,t)\|^2_{L^2(R^n)} \leq C(T)\{\|\varphi\|^2_{H^1(R^n)} + \|\psi\|^2_{L^2(R^n)}\} \tag{9}$$

for every $|t| \leq T$ and a positive constant $C(T)$ which is a linear combination of positive powers of T. Here, $\|.\|_{H^1(R^n)}$ and $\|.\|_{L^2(R^n)}$ denotes the usual norms of the spaces $H^1(R^n)$ and $L^2(R^n)$ respectively. Now, integrating (9) with respect to t in the interval $[-T,T]$ we obtain

$$\|w\|^2_{H^1(R^n \times [-T,T])} \leq \widetilde{C}(T)\{\|\varphi\|^2_{H^1(R^n)} + \|\psi\|^2_{L^2(R^n)}\} \tag{10}$$

where $\widetilde{C}(T)$ is also a linear combination of powers of T. Since $T > 0$ was taken arbitrarily and C_0^∞ is dense in both H^1 and L^2, the estimate (10) allows us to set up the following definition.

Definition 2.1: *Given $(u_0, u_1) \in H^1(R^n) \times L^1(R^n)$, a function $u \in H^1_{loc}(R^n \times R)$ is the solution of the Cauchy problem for the homogeneous wave equation with initial data (u_0, u_1) if there exists a sequence $(w_k)_{k=1}^\infty \in C^\infty(R^n \times R)$ of solutions of the Cauchy problem (6)-(8) with initial data $(\varphi_k, \psi_k) \in C_0^\infty(R^n) \times C_0^\infty(R^n)$ such that $\lim\limits_{k\to\infty}(\varphi_k, \psi_k) = (u_0, u_1)$ in $H^1(R^n) \times L^1(R^n)$ and $\lim\limits_{k\to\infty} w_k = u$ in $H^1(K)$ for every compact $K \subset R^n \times R$.*

Observe that the estimate (10) assures the well-posedness of the Cauchy problem for the homogeneous wave equation with initial data $(u_0, u_1) \in H^1(R^n) \times L^1(R^n)$ as in definition 2.1 while estimate (9) defines traces $(u(.,t), u_t(.,t)) \in H^1(R^n) \times L^2(R^2)$ for every $t \in R$.

We start discussing the local energy decay by establishing the following:

Lemma 2.2: *Let $U \subset R^2$ be a bounded domain and φ, $\psi \in C_o^\infty(R^2)$ be functions with compact support contained in U. Let $w \in C^\infty(R^2 \times R)$ be the solution of the Cauchy problem (6)-(8) with initial data (φ, ψ). For each $T_o > diam(U)$ there exists $K = K(T_o, U) > 0$ such that for each multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in N^3$, with $|\alpha| = \alpha_1 + \alpha_2 + \alpha_2 \leq 1$ we have*

$$\left|\frac{\partial^{|\alpha|}}{\partial(x,t)^\alpha} w(x,t)\right| \leq \frac{K}{t}\{\|\varphi\|_{H^1(U)} + \|\psi\|_{L^2(U)}\} \tag{11}$$

for every $x \in U$ and $t \geq T_o$.

Proof: Function w is given, for every $t > 0$, by the formula

$$w(x,t) = \frac{1}{2\pi}\int_{r<t} \frac{\psi(y)}{\sqrt{t^2 - r^2}}dy + \frac{\partial}{\partial t}[\frac{1}{2\pi}\int_{r<t} \frac{\varphi(y)}{\sqrt{t^2 - r^2}}dy] \tag{12}$$

where $r = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$, (see John (1986)). The domain of integration in (12) is the ball of center x and radius t. For each $x \in U$ and $t \geq T_o$ this ball includes U in its interior. Since the initial data has its support in U, we may change the domain of integration in (12) by U and write

$$w(x,t) = \frac{1}{2\pi}\int_U \frac{\psi(y)}{\sqrt{t^2 - r^2}}dy + \frac{\partial}{\partial t}[\frac{1}{2\pi}\int_U \frac{\varphi(y)}{\sqrt{t^2 - r^2}}dy] \tag{13}$$

for every $t \geq T_o$.

In order to estimate all the derivatives of w listed in (11) it suffices to estimate the function $\zeta(x,y,t) = \frac{1}{\sqrt{t^2 - r^2}}$ and its derivatives $\frac{\partial}{\partial x_1}$, $\frac{\partial}{\partial x_2}$, $\frac{\partial}{\partial t}$, $\frac{\partial^2}{\partial x_1 \partial t}$, $\frac{\partial^2}{\partial x_2 \partial t}$ and $\frac{\partial^2}{\partial t^2}$, uniformly for $x, y \in U$ and $t > T_o$. The symmetry between x_1 and x_2 reduces the work to estimate only ζ, $\frac{\partial \zeta}{\partial x_1}$, $\frac{\partial \zeta}{\partial t}$, $\frac{\partial^2 \zeta}{\partial x_1 \partial t}$ and $\frac{\partial^2 \zeta}{\partial t^2}$.

Observe that for $x, y \in U$ and $t \geq T_o$ we have $\frac{r}{t} \leq \frac{diam(U)}{T_o} < 1$. Now, let $\lambda > 0$ be so that $\frac{diam(U)}{T_o} < \lambda < 1$. If we define $\varkappa(s) = \frac{1}{\sqrt{1 - s^2}}$ we see that $\varkappa$ is a C^∞ function on the compact interval $[-\lambda,\ \lambda]$. Hence, there exists a constant $K_1 > 0$ such that

$$\varkappa(\tfrac{r}{t})^k \leq \ K_1,$$

for every $x, y \in U$, $t > T_o$ and $k = 1, 2, ..., 5$. .

A direct calculation gives $\zeta(x,y,t) = \frac{1}{t}\varkappa(\frac{r}{t})$, $\frac{\partial \zeta}{\partial x_1}(x,y,t) = \frac{1}{t}[\frac{x_1 - y_1}{t^2}\varkappa(\frac{r}{t})^3]$, $\frac{\partial \zeta}{\partial t}(x,y,t) = \frac{1}{t}[\frac{-1}{t}\varkappa(\frac{r}{t})^3]$, $\frac{\partial^2 \zeta}{\partial x_1 \partial t}(x,y,t) = \frac{1}{t}[\frac{-3(x_1 - y_1)}{t^3}\varkappa(\frac{r}{t})^5]$ and $\frac{\partial^2 \zeta}{\partial t^2}(x,y,t) = \frac{1}{t}[\frac{3}{t^2}\varkappa(\frac{r}{t})^5 - \frac{1}{t^2}\varkappa(\frac{r}{t})^3]$. Observe that the expressions inside each of the brackets above are bounded by a constant $K_2 > 0$ (depending on K_1, T_o and $diam(U)$), uniformly for $x, y \in U$ and $t \geq T_o$. Hence, the absolute value of ζ and its derivatives are uniformly bounded with respect to $x, y \in U$ by $\frac{K_2}{t}$ for every $t \geq T_o$.

Now, for each $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, with $|\alpha| \leq 1$ we differentiate (13) and take the absolute value of both sides. After using the estimates just obtained for ζ and its derivatives we get

$$\left| \frac{\partial^{|\alpha|}}{\partial(x,t)^\alpha} w(x,t) \right| \leq \frac{K_2}{2\pi t}\{\int_U |\psi(y)|\, dy + \int_U |\varphi(y)|\, dy\}$$

for every $x \in U$ and $t \geq T_o$. Now, using the Cauchy-Schwartz inequality and $\|\varphi\|_{L^2} \leq \|\varphi\|_{H^1}$ in the last inequality we obtain

$$\left| \frac{\partial^{|\alpha|}}{\partial(x,t)^\alpha} w(x,t) \right| \leq \frac{K_2 |U|^{1/2}}{2\pi t}\{\|\varphi\|_{H^1(U)} + \|\psi\|_{L^2(U)}\}$$

where$|U|$ is the Lebesgue measure of U, for every $x \in U$ and $t \geq T_o$. Now, setting $K = \frac{K_2 |U|^{1/2}}{2\pi}$ we obtain (11). ■

In order to obtain an appropriate decay estimate to solve the controllability problem we work a little more on estimate (11). First we square both sides of (11), then use the elementary inequality $(a+b)^2 \le 2(a^2+b^2)$ and integrate over U to obtain : $\left\| \frac{\partial^{|\alpha|}}{\partial(x,t)^\alpha} w(.,t) \right\|^2_{L^2(U)} \le \frac{2K^2|U|}{t^2}\{\|\varphi\|^2_{H^1(U)} + \|\psi\|^2_{L^2(U)}\}$ for every $|\alpha| \le 1$ and $t \ge T_o$. Now, summing up the last inequality for the appropriate α with $|\alpha| \le 1$ we obtain

$$\|w(.,t)\|^2_{H^1(U)} + \|w_t(.,t)\|^2_{L^2(U)} \le \tfrac{M}{t^2}\{\|\varphi\|^2_{H^1(U)} + \|\psi\|^2_{L^2(U)}\} \tag{14}$$

where $M > 0$ depends only on T_o and U, for every $t \ge T_o$.

Before we state the next theorem we observe that for every open set $U \subset R^2$, the elements of $H^1_0(U)$ may be seen as functions of $H^1(R^2)$ vanishing outside U (see Dautray and Lions (1988), p.118).

Theorem 2.3: *Let $U \subset R^2$ be a bounded domain, $(u_0, u_1) \in H^1_0(U) \times L^2(U)$, $(\widetilde{u}_0, \widetilde{u}_1) \in H^1(R^2) \times L^2(R^2)$ be the extension of (u_0, u_1) by zero outside U, and $T_o > diam(U)$. Then there exists $M > 0$, depending only on U and T_o such that the solution $u \in H^1_{loc}(R^2 \times R)$ of the Cauchy problem for the homogeneous wave equation with initial data $(\widetilde{u}_0, \widetilde{u}_1)$ satisfies*

$$\|u(.,t)\|^2_{H^1(U)} + \|u_t(.,t)\|^2_{L^2(U)} \le \tfrac{M}{t^2}\{\|u_0\|^2_{H^1(U)} + \|u_1\|^2_{L^2(U)}\} \tag{15}$$

for every $t \ge T_o$.

Proof: Take $(w_k)_{k=1}^\infty$, $(\varphi_k)_{k=1}^\infty$ and $(\psi_k)_{k=1}^\infty$ as in definition 2.1 with suppφ_k, supp$\psi_k \subset U$ for every k. Apply (14) to the triple (φ_k, ψ_k, w_k) and take the limit as $k \to \infty$ to obtain (15). ∎

In Russell (1973) an estimate like (15) was obtained for solutions $u \in H^m_{loc}(R^n \times R)$ of the homogeneous wave equation, with $m \ge 2$ and n even.

We finish this section setting a result on the regularity of the traces of the solutions of the wave equation which will be needed to study the controllability problem. We start explaining Tataru's results (see Tataru (1998)). Let $P(\xi, D)$ be a linear second order hyperbolic partial differential operator with C^∞ coefficients depending on ξ in some open bounded domain $\Xi \subset R^n$. Let $\Sigma \subset \Xi$ be an oriented C^∞ hypersurface, noncharacteristic time-like with respect to $P(\xi, D)$. Let $\nu = (\nu_1, ...\nu_n)$ be the unit vector normal to Σ. If $\Sigma a^{ij} \frac{\partial^2}{\partial\xi_i \partial\xi_j}$ is the principal part of $P(\xi, D)$, then $\frac{\partial u}{\partial\nu} = \Sigma a^{ij} \frac{\partial u}{\partial\xi_i}\nu_j$ is the conormal derivative of u with respect to $P(\xi, D)$ along Σ. In Tataru (1998), Theorem 2 proves that if $u \in H^1_{loc}(\Xi)$ is such that $P(\xi, D)u \in L^2_{loc}(\Xi)$ then $\frac{\partial u}{\partial\nu} \in L^2_{loc}(\Sigma)$. Now we apply this result to the wave equation. Let $\Xi = R^3 = R^2 \times R$ and $\Sigma = \gamma \times R$ where γ is a C^∞curve in R^2 with no self intersection and normal vector $\nu = (\nu_1, \nu_2)$. Taking $P = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1} - \frac{\partial^2}{\partial x_2}$ and observing that $(\nu_1, \nu_2, 0)$ is the normal vector to Σ we see that the conormal derivative of a function u, with respect to P, along Σ is merely the usual normal derivative $\frac{\partial u}{\partial\nu}$. The following theorem is an immediate consequence of Tataru's theorem and the usual trace theorems (see Dautray and Lions (1988)).

Theorem 2.4: *Let $u \in H^1_{loc}(R^2 \times R)$ be the solution of the homogeneous wave equation with initial data $(u_0, u_1) \in H^1(R^2) \times L^2(R^2)$. Let γ be a C^∞ curve in R^2 with no self intersection and normal vector ν. Then the normal derivative of u along $\gamma \times R$ has trace $\frac{\partial u}{\partial \nu} \in L^2_{loc}(\gamma \times R)$. Moreover, $\alpha u + \beta \frac{\partial u}{\partial \nu} \in L^2_{loc}(\gamma \times R)$ for every $\alpha, \beta \in R$.*

3 The main theorem

Let Ω be a curved polygon, $\delta > 0$, $\Omega_\delta = \{y \in R^2 : \exists\, x \in \Omega,\ |x - y| < \delta\}$ an open neighborhood of Ω and $T_o >\ diam(\ \Omega_\delta)$. Given $(w_0, w_1) \in H^1_0(\Omega_\delta) \times L^2(\Omega_\delta)$ we can regard w_0 and $w_1$as functions of $H^1(R^2)$ and $L^2(R^2)$ vanishing outside Ω_δ. For convenience we will not distinguish between $(w_0, w_1) \in H^1_0(\Omega_\delta) \times L^2(\Omega_\delta)$ and its extension $(w_0, w_1) \in H^1_0(R^2) \times L^2(R^2)$, by zero outside Ω_δ.

Given $(w_0, w_1) \in H^1_0(\Omega_\delta) \times L^2(\Omega_\delta)$ let $w \in H^1_{loc}(R^2 \times R)$ be the solution of the Cauchy problem;

$$w_{tt} - \Delta w = 0 \qquad in\ \ R^2 \times R \tag{16}$$
$$w(x, 0) = w_0(x) \qquad in\ R^2 \tag{17}$$
$$w_t(x, 0) = w_1(x) \qquad in\ R^2. \tag{18}$$

For each $T > T_o$ we define the linear operator

$$\begin{array}{ll} S_T: & H^1_0(\Omega_\delta) \times L^2(\Omega_\delta) \to H^1(R^2) \times L^2(R^2) \\ & \qquad (w_0, w_1) \quad \to \quad (w(., T), w_t(., T)) \end{array}$$

where w is the solution of (16)-(18).

The decay estimate (15) with $U = \Omega_\delta$ applied to S_T furnishes

$$\|w(., T)\|^2_{H^1(\Omega_\delta\ )} + \|w_t(., T)\|^2_{L^2(\Omega_\delta\ )} \le \frac{M}{T^2}\{\|w_0\|^2_{H^1(\Omega_\delta\ )} + \|w_1\|^2_{L^2(\Omega_\delta\ )}\} \tag{19}$$

for every $T\ >\ T_o$, where the constant M depends only on Ω_δ and T_o.

Now we consider a backward Cauchy problem for the wave equation. Let $(z_0, z_1) \in H^1_0(\Omega_\delta) \times L^2(\Omega_\delta)$ and $z \in H^1_{loc}(R^2 \times R)$ be the solution of the Cauchy problem

$$z_{tt} - \Delta z = 0 \qquad in\ R^2 \times R \tag{20}$$
$$z(x, T) = z_0(x) \qquad in\ R^2 \tag{21}$$
$$z_t(x, T) = z_1(x) \qquad in\ R^2 \tag{22}$$

Observe that the function $W \in H^1_{loc}(R^2 \times R)$, $W = W(x, \tau)$ defined by $W(x, \tau) = z(x, T - \tau)$ solves the Cauchy problem (16)-(18) with initial data $(z_0, -z_1)$ (at $\tau = 0$). Hence, using (19) we get $\|W(., T)\|^2_{H^1(\Omega_\delta\ )} + \|W_\tau(., T)\|^2_{L^2(\Omega_\delta\ )} \le \frac{M}{T^2}\{\|z_0\|^2_{H^1(\Omega_\delta\ )} + \|-z_1\|^2_{L^2(\Omega_\delta\ )}\}$, which turns out to be

$$\|z(., 0)\|^2_{H^1(\Omega_\delta\ )} + \|z_t(., 0)\|^2_{L^2(\Omega_\delta\ )} \le \frac{M}{T^2}\{\|z_0\|^2_{H^1(\Omega_\delta\ )} + \|z_1\|^2_{L^2(\Omega_\delta\ )}\}, \tag{23}$$

for every $T\ >\ T_o$.

Now we define the linear operator

$$S_T^*: \ H_0^1(\Omega_\delta) \times L^2(\Omega_\delta) \to H^1(R^2) \times L^2(R^2)$$
$$(z_0, z_1) \quad \to \quad (z(.,0), z_t(.,0))$$

where z is the solution of (20)-(22). Estimates (19) and (23) show the boundedness of S_T and S_T^* respectively. Now we state and prove the main result of the paper.

Theorem 3.1 : *Given a curved polygon* Ω, *there exists* $T > diam(\Omega)$ *so that, for every initial state* $(u_0, u_1) \in H^1(\Omega) \times L^2(\Omega)$ *there exists a control function* $g \in L^2(\Gamma \times [0,T])$ *such that the solution of the system*

$$\begin{array}{ll} u_{tt} - \Delta u = 0 & in\ \Omega \times [0,T] \\ u(x,0) = u_0(x) & in\ \Omega \\ u_t(x,0) = u_1(x) & in\ \Omega \\ \alpha u + \beta \frac{\partial u}{\partial \nu} = g & on\ \Gamma \times [0,T] \end{array}$$

satisfies the final condition

$$u(x,T) = \frac{\partial u}{\partial \nu}(x,T) = 0 \qquad in\ \Omega.$$

Proof : The idea of the proof is to construct an extension $(\widetilde{u}_0, \widetilde{u}_1) \in H^1(R^2) \times L^2(R^2)$ of the initial data with support in Ω_δ, such that the solution $\widetilde{u} \in H^1_{loc}(R^2 \times R)$ of the Cauchy problem for the homogeneous wave equation with initial data $(\widetilde{u}_0, \widetilde{u}_1)$, at time $t = 0$, satisfies $(\widetilde{u}(.,T), \widetilde{u}_t(.,T)) = (0,0)$ for some $T > diam(\Omega)$. Once it is done, we define u as the restriction of $\widetilde{u}$ to the finite cylinder $\Omega \times [0,T]$, use theorem 2.4 to read off the trace $\alpha\widetilde{u} + \beta\frac{\partial \widetilde{u}}{\partial \nu}$ as a function of $L^2(\Gamma \times [0,T])$ and define the control g by $g = \alpha\widetilde{u} + \beta\frac{\partial \widetilde{u}}{\partial \nu}$. The just found $T > diam(\Omega)$ and $g \in L^2(\Gamma \times [0,T])$ solve the controllability problem. Note that theorem 2.4 is applied to each smooth part $\gamma \times [0,T]$ of $\Gamma \times [0,T]$, then we paste together the finitely many functions (traces) obtained to build the control g.

To complete the proof it remains only to find the appropriate $T > 0$ and prove the existence of the mentioned extension $(\widetilde{u}_0, \widetilde{u}_1)$ of the initial state (u_0, u_1). From now on we focus on this question. Fix $\delta > 0$, let Ω_δ be the open neighborhood of Ω as in the beginning of this section and fix $T_o > diam(\Omega_\delta)$. Let $E_0 : H^1(\Omega) \to H^1(R^2)$ be the bounded linear operator which extends $w_0 \in H^1(\Omega)$ to a function $E_0(w_0) \in H^1(R^2)$ with compact support in Ω_δ. The existence of operator E_0 follows from the fact that Ω satisfies the uniform cone condition and the Calderon-Zigmund Theorem (see Wrokla (1987), p. 95). The continuity of E_0 is expressed by the estimate $\|E_0(w_0)\|^2_{H^1(\Omega_\delta)} = \|E_0(w_0)\|^2_{H^1(R^2)} \leq c_0 \|w_0\|^2_{H^1(\Omega)}$ for some $c_0 > 0$ and any $w_0 \in H^1(\Omega)$. Let $E_1 : L^2(\Omega) \to L^2(R^2)$ be the bounded linear operator which extends $w_1 \in L^2(\Omega)$ to a function $E_1(w_1) \in L^2(R^2)$ vanishing outside Ω. We have $\|E_1(w_1)\|^2_{L^2(\Omega_\delta)} = \|E_1(w_1)\|^2_{L^2(R^2)} = \|w_1\|^2_{L^2(\Omega)}$. We will denote E the bounded linear operator given by $E = (E_0, E_1)$ from $H^1(\Omega) \times L^2(\Omega)$ into $H^1(R^2) \times L^2(R^2)$.

Let $(w_0, w_1) \in H^1(\Omega) \times L^2(\Omega)$ be an arbitrary initial state and $w \in H^1_{loc}(R^2 \times R)$ be the solution of the problem

$$\begin{array}{ll} w_{tt} - \Delta w = 0 & in\ R^2 \times R \\ w(x,0) = (E_0 w_0)(x) & in\ R^2 \end{array}$$

$$w_t(x,0) = (E_1 w_1)(x) \qquad in\ R^2$$

Let $T > T_o$ be a real number to be chosen later and $\theta \in C_0^\infty\left(R^2\right)$ a cut off function such that $\theta \equiv 1$ on $\Omega_{\delta/2}$ and $\theta \equiv 0$ outside Ω_δ. Let $z \in H^1_{loc}(R^2 \times R)$ be the solution of the backward Cauchy problem

$$\begin{array}{ll} z_{tt} - \Delta z = 0 & in\ \ R^2 \times R \\ z(x,T) = \theta\,(x)\, w(x,T) & in\ R^2 \\ z_t(x,T) = \theta\,(x)\, w_t(x,T) & in\ R^2 \end{array}$$

Now, consider the function $\widetilde{u} = w - z \in H^1_{loc}(R^2 \times R)$ and observe that $\widetilde{u}$ satisfies

$$\begin{array}{ll} \widetilde{u}_{tt} - \Delta \widetilde{u} = 0 & in\ \ R^2 \times R \\ \widetilde{u}(x,0) = (E_0 w_0)(x) - z\,(x,0) & in\ R^2 \\ \widetilde{u}_t(x,0) = (E_1 w_1)(x) - z_t(x,0) & in\ R^2 \end{array}$$

and

$$\begin{array}{l} \widetilde{u}(x,T) = w(x,T) - \theta\,(x)\, w(x,T) = 0\ ,\ x \in \Omega_{\delta/2} \\ \widetilde{u}_t(x,T) = w_t(x,T) - \theta\,(x)\, w_t(x,T) = 0\ ,\ x \in \Omega_{\delta/2} \end{array}$$

since $\theta \equiv 1$ in $\Omega_{\delta/2}$. Hence $\widetilde{u}$ solves the homogeneous wave equation and has the desirable final state $(\widetilde{u}(.,T), \widetilde{u}_t(.,T)) = (0,0)$ in Ω. According to the comment in the beginning of the proof, the job would be done if we knew the value of T and if $(\widetilde{u}(.,0), \widetilde{u}_t(.,0))$ extended the initial data (u_0, u_1).

Consider the equalities

$$\widetilde{u}(x,0) = (E_0 w_0)(x) - z\,(x,0) = u_0\,(x), \tag{24}$$

$$\widetilde{u}_t(x,0) = (E_1 w_1)(x) - z_t(x,0) = u_1(x),. \tag{25}$$

in Ω. We want to solve (24)-(25) for $(w_0, w_1) \in H^1\,(\Omega) \times L^2\,(\Omega)$. In order to do it, we reconstruct equations (24)-(25) in terms of operators S_T, S_T^* and the operator multiplication by θ. Note that:

$$(z(.,0)\,, z_t\,(.,0)) = S_T^*\,(\theta w(.,T), \theta w_t(.,T)) = S_T^* \theta [S_T\,(E_0 w_0, E_1 w_1)] = [S_T^* \theta S_T E]\,(w_0, w_1)$$

and then (24)-(25) turns into

$$\{E\,(w_0, w_1) - [S_T^* \theta S_T E]\,(w_0, w_1)\}\,(x) = (u_0(x), u_1(x)), \quad x \in \Omega.$$

which can be rewritten as

$$[Id - \Re S_T^* \theta S_T E]\,(w_0, w_1) = (u_0, u_1)$$

where Id is the identity in $H^1\,(\Omega) \times L^2\,(\Omega)$ and $\Re$ is the restriction to Ω.

At this point, to complete the proof we use the energy decay estimates (19) and (23) to prove that there exists $T > 0$ such that $K_T \equiv \Re S_T^* \theta S_T E$ is a contraction in $H^1\,(\Omega) \times L^2\,(\Omega)$. Observe that

$$\|\Re S_T^* \theta S_T E\,(w_0, w_1)\|^2_{H^1(\Omega)\times L^2(\Omega)} = \|(z(.,0)\,, z_t\,(.,0)\|^2_{H^1(\Omega)\times L^2(\Omega)} \leq$$
$$\leq \|(z(.,0)\,, z_t\,(.,0)\|^2_{H^1(\Omega_\delta)\times L^2(\Omega_\delta)} \leq \tfrac{M}{T^2}\,\|(z(.,T)\,, z_t\,(.,T)\|^2_{H^1(\Omega_\delta)\times L^2(\Omega_\delta)} =$$
$$= \tfrac{M}{T^2}\,\|(\theta w(.,T)\,, \theta w_t\,(.,T)\|^2_{H^1(\Omega_\delta)\times L^2(\Omega_\delta)} \leq \tfrac{\widetilde{M}}{T^2}\,\|(w(.,T)\,, w_t\,(.,T)\|^2_{H^1(\Omega_\delta)\times L^2(\Omega_\delta)} =$$

$= \frac{\widetilde{M}}{T^2} \|S_T E(w_0, w_1)\|^2_{H^1(\Omega_\delta)\times L^2(\Omega_\delta)} \leq \frac{M\widetilde{M}}{T^4} \|E(w_0, w_1)\|^2_{H^1(\Omega_\delta)\times L^2(\Omega_\delta)} \leq$
$\leq \frac{M\widetilde{M}(c_0+1)}{T^4} \|(w_0, w_1)\|^2_{H^1(\Omega)\times L^2(\Omega)}.$

Hence, $\|K_T(w_0, w_1)\|^2_{H^1(\Omega)\times L^2(\Omega)} \leq \frac{M\widetilde{M}(c_0+1)}{T^4} \|(w_0, w_1)\|^2_{H^1(\Omega)\times L^2(\Omega)}$ for every $(w_0, w_1) \in H^1(\Omega) \times L^2(\Omega)$ and $T > T_o$. Since $M\widetilde{M}(c_0 + 1)$ does not depend on T, we choose $T > T_o$ so that $\sqrt{\frac{M\widetilde{M}(c_0+1)}{T^4}} < 1$ and conclude the proof.■

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■ *RESUMO: Neste trabalho estudamos a controlabilidade exata na fonteira para a equação da onda bidimesional em domínios denominados polígonos curvos. Os dados iniciais possuem energia finita e o controle obtido é do tipo Neumann e de quadrado integravel.*

■ *PALAVRAS-CHAVE: Controle exato na fronteira; equação da onda; controle do tipo Neumann; domínios não suaves.*

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