AN APPROXIMATE BAYESIAN ANALYSIS FOR ACCELERATED TESTS WITH LOG-NON-LINEAR STRESS-RESPONSE RELATIONSHIP

Gleici Castro PERDONÁ[1]
Francisco LOUZADA-NETO[2]

▪ ABSTRACT: In this paper we propose an accelerated life tests model with a log-non-linear stress-response relationship. The advantage of such a formulation is that the general framework accommodates several stress-response relationships usually considered in accelerated life tests, while it is flexible enough for fitting the data that cannot be accommodatet by a simple log-linear stress-response relationship. A Bayesian approach with constant prior density is considered for interval estimation by using Laplace's method to find the marginal posterior densities of interest. Model comparation is made through Bayes factors. The methodology is illustrated in a numerical example.

▪ KEYWORDS: Accelerated lifetime tests; Bayesian analysis; exponential distribution; Laplace method; log-non-linear stress-response relationship.

1 Introduction

Accelerated life tests (ALT) can be performed in engineering applications by testing items at higher stress covariate levels than the usual working conditions. Several authors have been working on ALT and there is a large amount of literature on this topic. Interested readers can refer to Mann, Schaffer and Singpurwalla (1974), Nelson (1990) and Meeker and Escobar (1998), which are three excellent sources for ALT.

A simple ALT scenario can be characterized by submitting k groups of n_i items each under k constant and fixed stress covariate levels, $X_1, ..., X_k$ (hereafter stress

[1]Departamento de Medicina Social, Faculdade de Medicina de Ribeirão Preto - FMRP, Universidade de São Paulo - USP, CEP 14049-900, Ribeirão Preto, SP, Brasil. E-mail: *pgleici@fmrp.usp.br*

[2]Departamento de Estatística, Universidade Federal de São Carlos - UFSCar, Caixa Postal 676, CEP 13565-905, São Carlos, SP, Brasil. E-mail: *dfln@power.ufscar.br*

levels). The experiment ends after the occurence of a certain pre-fixed number $r_i < n_i$ of failures, $t_{i1}, t_{i2}, ..., t_{ir_i}$, at each stress level, characterizing a type II censoring scheme (Lawless, 1982).

ALT models are composed of a probabilistic component, which is represented by a lifetime distribution, such as exponential, Weibull, log-normal, log-logistic, among others, and by a stress-response relationship (SRR), which relates the mean lifetime (or a function of this parameter) with stress levels. Common SRRs are the power law, Eyring and Arrhenius models, which are log-linear SRRs (Mann, Schaffer and Singpurwalla, 1974). In practice, however, it is not uncommon to find phenomena that cannot be accommodated by only a log-linear SRR. These situations arise, in particular, when even the insertion of a small stress can induce the immediate device failure (Nelson, 1990).

In this paper we propose an ALT with a log-non-linear SRR considering an exponential distribution. The advantage of such formulation is that the general framework accommodates several SRRs usually considered in ALTs, while it is flexible enough for fitting the data that cannot be accommodated by a simple log-linear SRR. We also show how Bayesian estimation procedures be adopted for an ALT with a log-non-linear SRR as an alternative to the frequentist method. The advantage of this procedure is that it leads to a single algorithm for fitting hazard-based models, and model comparation is easily made through Schwarz criterion and/or Bayes Factors. Due to the mathematical difficulties involved in the direct Bayesian estimation for parameters of the model, we use Laplace method (Tierney and Kadane, 1986) to find the marginal posterior densities of interest. In Section 2 we formulate the ALT model with a log-non-linear SRR and briefly discuss maximum likelihhod estimation procedures. The Bayesian estimation procedure and model comparation is detailed in Section 3. The methodology is illustrated with a data set in Section 4.

2 Model Formulation

Let T be the lifetime random variable with an exponential density

$$f(t, \lambda_i) = \lambda_i \exp\{-\lambda_i t\}, \tag{1}$$

where $\lambda_i > 0$ is an unknown parameter representing the constant failure rate for $i = 1, ..., k$ (number of stress levels).

The likelihood function for λ_i, under the i-th stress level X_i is given by

$$L_i(\lambda_i) = \left(\prod_{j=1}^{r_i} f(t_{ij}, \lambda_i)\right) (S(t_{ir_i}, \lambda_i))^{n_i - r_i} = \lambda_i^{r_i} \exp\{-\lambda_i A_i\}, \tag{2}$$

where $A_i = \sum_{j=1}^{r_i} t_{ij} + (n_i - r_i) t_{ir_i}$ denotes the total time in the test for the i-th stress level.

Considering data under k random stress levels, the likelihood function for the parameter vector $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$ is

$$L(\lambda) = \prod_{i=1}^{k} \lambda_i^{r_i} \exp\{-\lambda_i A_i\}. \tag{3}$$

The log-non-linear SRR is defined as,

$$\lambda_i = \exp(-\beta_0 - \beta_1 X_i^{\beta_2}) \tag{4}$$

where β_0, β_1 and β_2 are unknown parameters such as $-\infty < \beta_0$, β_1, $\beta_2 < \infty$, with parameter β_2 representing the log-non-linearity term.

SRR (4) has several models as particular cases. When $\beta_2 = 1$ we have a general log-linear SRR that accommodates the most common log-linear SRR used in reliability (Achcar and Louzada-Neto, 1992). Among others, we have the Arrehius model for $\beta_2 = 1$ and $Xi = 1/Vi$, $\beta_0 = -\alpha_1$ and $\beta_1 = \alpha_2$, where Vi denotes a level of the temperature variable. The power model is obtained for $\beta_2 = 1$ and $Xi = -log(Vi)$, $\beta_0 = log(\alpha)$ and $\beta_1 = \alpha_2$, where Vi denotes a level of the voltage variable (Nelson, 1990). Model (4) also allows an extra variability in the fitting. From the practical point of view, this means that it allows a more intense decrease or increase in the failure rate with small changes in covariate levels, which could be valuable for studies envolving very sensitive devices.

The likelihood function for β_0, β_1 and β_2 follows from (3) and (4) and is given by,

$$L(\beta_0, \beta_1, \beta_2) = \prod_{i=1}^{k} \{\exp(-\beta_0 - \beta_1 V_i^{\beta_2})^{r_i} \exp(-\exp(-\beta_0 - \beta_1 V_i^{\beta_2}) A_i)\}. \tag{5}$$

The maximum likelihood estimates (MLEs) of β_0, β_1 and β_2 can be obtained by direct maximization of (5) or by solving the system of nonlinear equations, $\partial \log L / \partial \varphi = 0$, where $\varphi' = (\beta_0, \beta_1, \beta_2)$. Obtaining the score function is conceptually simple and the expressions are not given explicitly.

Large-sample inference for the parameters can be based on MLEs and their estimated variances obtained by inverting the expected information matrix. For small or moderate-sized samples, however, confidence intervals for the parameters are probably best obtained using their profile log likelihoods, once large-sample approximations for likelihood ratio statistics are generally more reliable than those for MLEs. In order to avoid dependence on asymptotics approximations we can apply simulation approaches, such as the parametric bootstrap (Davison and Hinkley, 1997). As the methods involved are standard, we turn to Bayesian inference.

3 Bayesian estimation procedure

Define $\varphi = (\beta_0, \beta_1, \beta_2)$ as the parameter vector in the model which will be fitted. From the Bayesian point of view if there is lack of prior information about the parameters, a possibility is to assume a constant prior density for them with support over the real line (Box and Tiao, 1973). In this case, the joint posterior density $\pi(\varphi/data)$ is (5), apart from the normalizing constant. This procedure has the advantage of promoting a direct comparison between the frequentist and Bayesian approaches. The latter is briefly considered in the numerical example section.

Inferences about the parameters can be based on their marginal posterior densities, which can be obtained by integrating the joint posterior density. In our case it is not possible to obtain analytical solutions for the integrals. In order to solve this problem different existing strategies can be used. Particularly, numerical techniques that ranges from Simpson's rule to Monte Carlo-type and Gibbs sampling approaches (Naylor and Smith, 1982 and Gelfand and Smith, 1990) or approximation methods for integrals (Tierney and Kadane, 1986). The choice of the best strategy depends on the computational cost and the accuracy of the obtained results. We shall consider Laplace method for approximating integrals (Tierney and Kadane, 1986; Kass, Tierney and Kadane, 1990), which is a rather non-intensive computational procedure. As we will see in the numerical example section, this method has good accuracy in our application.

For instance, assume we are interested in inference about, say β_2, but their others parameters or functions could be considered. Using Laplace method to approximate the integrals in the marginal posterior density for β_2 (Tierney and Kadane, 1986; Kass, Tierney and Kadane, 1990), the approximate marginal posterior density for β_2 is

$$\pi(\beta_2/data) \propto (2\pi)^{d/2} \left|D^*_{-r}\right|^{-1/2} \pi(\varphi^*/data), \tag{6}$$

where d is the dimension of φ, $\varphi^* = (\hat{\beta}_0, \hat{\beta}_1, \beta_2)$ are MLEs of (β_0, β_1) with β_2 held fixed and D^*_{-r} is the determinant of the Hessian matrix of minus the joint posterior density of φ. The marginal posterior density (6) must be renormalized and evaluated over a grid of values of β_2.

When we have prior knowledge about the parameters, represented by density $\pi(\varphi)$, it can be incorporated into the analysis by multipling $\pi(\varphi)$ by (5). Trivial changes are necessary in (6).

Laplace method may lose accuracy if the parameter vector is high-dimensional (Tierney and Kadane, 1986; Kass, Tierney and Kadane, 1990), which is not our case. Further, in reliability, experiments with a reduced number of covariates are common. Besides, if the posterior density is seriously multi-modal, Laplace method may fail, but we have not found this a problem in any of the examples to which we have applied the approximation.

Models can be compared using the Schwarz criterion and/or Bayes factors (Kass and Raftery, 1995). Defining φ_1 and φ_2 as the parameter vector under two different models, say model 1 and model 2, respectively, the Schwarz criterion is

$$\log L(\hat{\varphi}_1) - \log L(\hat{\varphi}_2) - \frac{1}{2}(d_1 - d_2)\log n, \tag{7}$$

where $\hat{\varphi}_1$ and $\hat{\varphi}_2$ are MLEs under the two models, and d_1 and d_2 are the dimensions of φ_1 and φ_2, respectively. The Schwarz criterion can be viewed as a rough approximation to the logarithm of the Bayes factors (Kass and Raftery, 1995). According to the rough classification in Section 3.2 of Kass and Raftery paper, Bayes factors between 1 and 3 give no evidence, between 3 and 20 positive evidence, between 20 and 150 strong evidence, and bigger than 150 very strong evidence against model 1.

This comparison is crude, but more detailed calculations will depend on the choice of prior, which may be specified if genuine prior information about the parameters is available. Prior specification is, however, out of the scope of this paper. As follows, the advantage of this comparison procedure is that it can be applied straightforwardly as a Bayesian comparison procedure for scientific reporting.

4 A numerical example

As a numerical example, consider a generate dataset on an ALT performed at three levels of stress, $\mathbf{X} = (1, 1/2, 1/3)$, with 30 non-censored items at each level. We assume the exponential distribution (1) for the lifetimes with SRR (4), with parameter values $\beta_0 = 2$, $\beta_1 = 9$ and $\beta_2 = 3$.

Figure 1 shows the approximate marginal posterior densities for β_0, β_1 and β_2 using Laplace method (6) and assuming a constant prior density for the parameters over the real line. The posterior modes of β_0, β_1 and β_2 are -1.98, 8.51 and 3.04, and their 90% highest posterior density confidence intervals are $(-2.19, -1.77)$, $(7.79, 9.36)$ and $(2.32, 6.04)$, respectively.

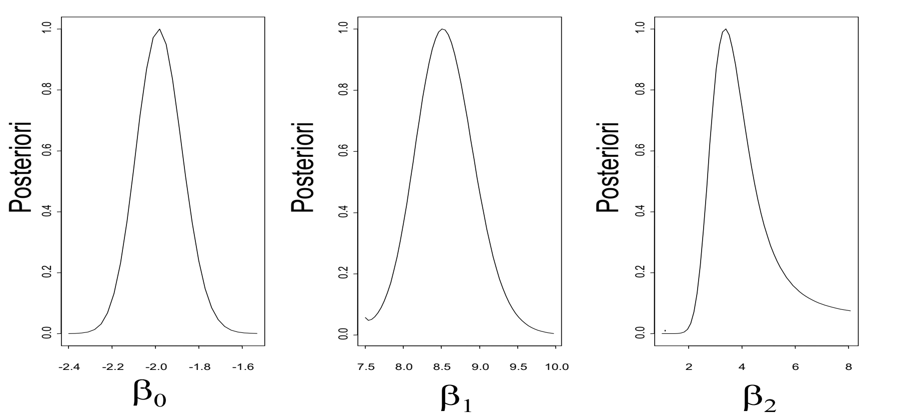

FIGURE 1 - Approximate marginal posterior densities for β_0, β_1 and β_2.

The Schwarz criterion (7) for comparison of the two models (exponential with log-linear SRR and log–non-linear SRR) is 37.9, which according to the rough classification in Section 3.2 of Kass and Raftery (1995) is 'very strong' evidence in favour of the full model.

For a limited comparison of the Bayesian results with the frequentist ones, we obtained the bootstrap percentile confidence intervals for the parameters generating $R = 999$ datasets from the exponential model with a log-non-linear SSR with the parameters substituted by their MLEs and recorded the MLEs of β_0, β_1 and β_2. For β_2 we obtained the ordened set $\widehat{\beta}^*_{2,1} < \widehat{\beta}^*_{2,2} < ... < \widehat{\beta}^*_{2,R}$, and used $\widehat{\beta}^*_{2,(R+1)(a/2)}$ and $\widehat{\beta}^*_{2,(R+1)(1-a/2)}$ as the lower and upper limits of the $100(1-a)\%$ confidence interval for β_2 (Davison and Hinkley, 1997). The same procedure was applied for obtaining the percentile confidence intervals for β_0 and β_1. The bootstrap percentile confidence intervals for β_0, β_1 and β_2 are $(-2.32, -1.81)$, $(8.04, 9.46)$ and $(2.48, 5.72)$, respectively. Interested readers can ask the authors for the R code developed for analysis.

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- *RESUMO: Neste trabalho consideramos um modelo de teste de sobrevivência acelerado com uma relação estresse-resposta log-não-linear. Esta formulação permite acomodar diversas relações estresse resposta usuais em testes de sobrevivência acelerado inclusive formas log-não-linear. Consideramos uma aproximação Bayesiana com priori constante para obtenção da estimação intervalar utilizando o método de Laplace para encontrarmos as densidades marginais a posteriori de interesse. Apresentamos a comparação dos modelos log-não-linear e log-linear utilizando o fator de Bayes e ilustramos a metodologia com um exemplo numérico.*

- *PALAVRAS-CHAVE: Testes de sobreviência acelerados; análise Bayesiana; distribuição exponencial, método de Laplace; relação estresse log-não-linear.*

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