EXPONENTIAL STABILITY FOR SEMIGROUP ASSOCIATED WITH A LINEAR VISCOELASTIC EQUATION BY A LOCALLY DISTRIBUTED DAMPING

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▪ *ABSTRACT: In this study we are concerned about the exponential stability of the semigroup associated with a linear viscoelastic wave equation with a locally distributed damping. We will use the method developed by LIU and ZHENG (1999). This method is very different from some others in the literature, such as the traditional energy method, see Rivera (1992), Kormonik's method, see Kormonik and Zuazua (1990) and Nakao's method, see Nakao (1986).*

▪ *KEYWORDS: Exponential decay; dissipative system; linear system; locally distributed damping.*

1 Introduction

Let us consider the motion of an elastic rod, $u(x,t)$, in the x direction with the reference configuration of lenght L. Suppose that the stress is of rate type, i. e.,

$$\sigma\alpha u_x + \gamma u_{xt}\ , \quad \text{where } \alpha > 0 \text{ and } \gamma > 0 \text{ are given constants.}$$

Then, the momentum equation is written as

$$u_{tt} - \alpha u_{xx} - \gamma u_{xxt} = 0,\ (x,t) \in (0,L) \times (0,\infty)\ .$$

If the rod is clamped at both ends, $x = 0$ and $x = L$, the boundary conditions $u(0,t) = u(L,t) = 0$ are imposed..

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Notice that γ being a constant in the equation above implies that the damping is distributed on the whole rod. In pratice, it is desirable to consider the problem with locally distributed damping (see Zuazua(1990)). Instead, we consider γ linear.

In this study we will consider the following initial and boundary value problem

$$
\begin{aligned}
u_{tt} - \alpha u_{xx} - \gamma(x)u_{xxt} &= 0, \ (x,t) \in (0,L) \times (0,\infty) && (1.1)\\
u(0,t) &= u(L,t) = 0, \ t > 0 && (1.2)\\
u(x,0) &= u_0(x) && (1.3)\\
u_t(x,0) &= u_1(x) && (1.4)
\end{aligned}
$$

where $\gamma(x) = ax + b$ with $a, b > 0$ and b will be chosen later.

The problem with dissipation of type $\gamma(x)$ constant was studied by Zheng-Liu (see Lui and Zheng(1999)). They proved the Exponential Stability for Semigroup Associated with a Linear Viscoelastic Equation. We will prove that the result is also valid whn $\gamma(x)$ is a linear function.

We conclude this section by collecting some results in the literature concerning the necessary and sufficient conditions for a C_0-semigroup being exponentially stable. The first result we will state is a necessary and sufficient condition for the exponential stability of a C_0-semigroup on a Hilbert space. It was obtained by Gearhart (see Wyler (1994) and Huang (1985)), independently. The following statement is due to Huang.

Theorem 1.1. Let $S(t) = e^{At}$ be a C_0-semigroup on a Hilbert space. Then $S(t)$ is exponentially stable if and only if

$$\sup\{Re\lambda; \lambda \in \sigma(A)\} \leq 0$$

and

$$\sup_{Re\lambda \geq 0} \|(\lambda I - A)^{-1}\| < \infty.$$

The following version of this result is due to Gearhart.

Theorem 1.2. Let $S(t) = e^{At}$ be a C_0-semigroup of contractions on a Hilbert space and $\rho(A)$ be the resolvent set of A. Then $S(t)$ is exponentially stable if and only if

$$\rho(A) \supseteq \{i\beta, \beta \in \mathbb{R}\}$$

and

$$\limsup_{|\beta| \to \infty} \|(i\beta - A)^{-1}\| < \infty.$$

Liu and Zheng have given the proof of the equivalence of these two results under the assumption that $S(t) = e^{At}$ is a C_0-semigroup of contractions on a Hilbert Space.

2 Existence and uniqueness of solution

The energy space associated with the system (1.1)-(1.4) is

$$H = H_0^1(0,L)\times L^2(0,L).$$

The inner product in the energy space is defined by $U_j(u^j, v^j)\in H,\ j=1,2,$ as follows:

$$(U_1,U_2)=\int_0^L \alpha u_x^1 v_x^1\,dx+\int_0^L u^2v^2\,dx.$$

Next, we will denote the norm in the energy space by $\|U\|^2=(U,U)$.

The system (1.1)-(1.4) can be written as

$$\frac{d}{dt}U(t)-AU(t)=0,$$

where

$$U=(u,v),\quad v=u_t,\quad \text{and}\quad A(u,v)=(v\,,(\alpha u_x+\gamma(x)v_x)_x-\alpha v_x)$$

with

$$D(A)=\{(u,v)\in H\,|\,v\in H_0^1,\ \alpha u_x+\gamma(x)v_x\in H^1\}.$$

Proposition 2.1. Operator A is dissipative.

Proof: Using the inner product we have

$$\begin{aligned}(AU,U) &= \int_0^L \alpha v_x u_x\,dx+\int_0^L(\alpha u_{xx}+\gamma(x)v_{xx})v\,dx\\ &= \int_0^L \alpha v_x u_x\,dx+\int_0^L \alpha u_{xx}v\,dx+\int_0^L\gamma(x)v_{xx}v\,dx.\end{aligned}$$

Performing integration by parts and using $v(0)=v(L)=0$ we have

$$\begin{aligned}(AU,U) &= \int_0^L \gamma(x)v_{xx}v\,dx\\ &= \int_0^L (ax+b)v_{xx}v\,dx\\ &= \int_0^L axv_{xx}v\,dx+b\int_0^L v_{xx}v\,dx.\end{aligned}$$

Again performing integration by parts and using $v(0)=v(L)=0$ we have

$$\begin{aligned}(AU,U) &= a\int_0^L v_xv\,dx+a\int_0^L x|v_x|^2\,dx-b\int_0^L|v_x|^2\,dx\\ &\le \frac{a}{2}\int_0^L|v_x|^2\,dx+\frac{a}{2}\int_0^L|v|^2\,dx+aL\int_0^L|v_x|^2\,dx-b\int_0^L|v_x|^2\,dx.\end{aligned}$$

Using Poincaré's inequality

$$(AU,U) \quad \leq \quad \frac{a}{2}\int_0^L |v_x|^2\,dx + \frac{aC_p}{2}\int_0^L |v_x|^2\,dx + aL\int_0^L |v_x|^2\,dx - b\int_0^L |v_x|^2\,dx.$$

Finally choosing b such that

$$\frac{a}{2} + \frac{aC_p}{2} + aL < b$$

we have

$$(AU,U) \quad \leq \quad -C\int_0^L |v_x|^2\,dx. \quad \blacksquare \tag{2.1}$$

Now for the sake of the reader, we collect the well-known Lumer-Phillips Theorem and its corollary concerning the generation of the C_0-semigroup generated by a dissipative operator.

Theorem 2.1. Let A be a linear operator with dense domain $D(A)$ in a Hilbert space H. If A is dissipative and there is a $\lambda_0 > 0$ such that the range, $R(\lambda_0 I - A)$, of $\lambda_0 I - A$ is H, then A is the infinitesimal generator of a C_0-semigroup of contractions on H.

As a corollary of the above theorem, the following result will be used in this paper.

Theorem 2.2. Let A be a linear operator with dense domain $D(A)$ in a Hilbert space H. If A is dissipative and $0 \in \rho(A)$, the resolvent set of A, then A is the infinitesimal generator of a C_0-semigroup of contractions on H.

Proof: By assumption $0 \in \rho(A)$, A is invertible and A^{-1} is a bounded linear operator. By the contraction mapping theorem, it is easy to see that operator $\lambda I - A = A(\lambda A^{-1} - I)$ is invertible for $0 < \lambda < ||A^{-1}||$. Therefore, it follows from the above Lumer-Phillips Theorem that A is the infinitesimal generator of a C_0-semogroup. of contractions on H. Thus the proof is complete. $\blacksquare$

Now we can prove the following.

Proposition 2.2. Operator A generates a C_0-semigroup of contractions $(e^{At})_{t>0}$ on H.

Proof: It is clear that $D(A)$ is dense on H and by Proposition 2.1 A is dissipative. We now prove that $0 \in (A)$. For any $F = (f, g) \in H$, consider the equation $AU = F$, i. e.,

$$v = f \in H_0^1 \tag{2.2}$$

$$\alpha u_{xx} + \gamma(x)v_{xx} = g \in L^2\ . \tag{2.3}$$

Using (2.2) in (2.3) we get

$$\alpha u_{xx} = g - \gamma(x) f_{xx} \in H^{-1} \ , \qquad (2.4)$$

where H^{-1} is the dual space of H_0^1. Thus, by a standard result in the theory of elliptic equations, we conclude that (2.4) admits a unique solution $u \in H_0^1$. It turns out tha we obtain a unique $(u, v) \in H$ such that $(u, v) \in D(A)$ and satisfies (2.2) and (2.3). It is clear that $||(u, v)||_H \leq K||F||_H$ with K being a positive constant. Thus $0 \in (A)$ is proven. By Theorem 2.2, the proof is complete. ∎

By the Theory of Semigroups, (see Pazy(1983)), it follows that

$$U(t) = e^{At}U_0, \quad U_0 = (u_0, v_0)$$

is the unique solution of

$$\frac{d}{dt}U(t) - AU(t) = 0$$

$$U(0) = U_0$$

and $U \in C^0(\ (0, \infty), D(A)\ ) \cap C^1(\ (0, \infty), H\ )$.

3 Exponential stability

The following theorem is the main result of the present work.

Theorem 3.1. The C_0-semigroup of contractions $(e^{At})_{t>0}$, generated by A, is exponentially stable, i. e., there exists positive constants M and w such that

$$||S(t)|| \leq Me^{-wt}.$$

Proof: We will use Theorem 1.2. First, by a contradiction argument, we will prove that

$$\rho(A) \supseteq \{i\beta, \beta \in \mathbb{R}\}. \qquad (3.1)$$

It follows from fact $0 \in \rho(A)$ and the contraction mapping theorem that for any real number β with $|\beta| < ||A^{-1}||^{-1}$, the operator $i\beta I - A = A(i\beta A^{-1} - I)$ is invertible. Moreover, $||(i\beta I - A)^{-1}||$ is a continuous function of β in the interval $(-||A^{-1}||^{-1}, ||A^{-1}||^{-1})$.

If $sup\,\{||(i\beta I - A)^{-1}|| \mid |\beta| < ||A^{-1}||^{-1}\} = M < \infty$, then by the contraction mapping theorem, the operator $i\beta I - A = (i\beta_0 I - A)(I + i(\beta - \beta_0)(i\beta_0 I - A)^{-1}$ with $|\beta_0| < ||A^{-1}||^{-1}$ is invertible for $|\beta - \beta_0| < \frac{1}{M}$. It turns out that by choosing $|\beta_0|$ as close to $||A^{-1}||^{-1}$ as we can, we conclude that $\{\beta \mid |\beta| <$

$||A^{-1}||^{-1} + \frac{1}{M}\} \subset \rho(A)$ and $||(i\beta I - A)^{-1}||$ is a continuous function of β in the interval $(-||A^{-1}||^{-1} - \frac{1}{M}, ||A^{-1}||^{-1} + \frac{1}{M})$.

Thus it follows from the argument above that if (3.2) is not true, then there is $\omega \in \mathbb{R}$ with $||A^{-1}||^{-1} \leq |\omega| < \infty$ such that $\{i\beta\, ;\, |\beta| < |\omega|\} \subset \rho(A)$ and $sup\{||(i\beta - A)^{-1}||\ |\ |\beta| < |\omega|\} = \infty$. It turns out that there exists a sequence $\beta_n \in \mathbb{R}$ with $\beta_n \to \omega$, $|\beta_n| < |\omega|$ and a sequence of complex vector functions $U_n = (u_n, v_n) \in D(A)$ with $||U_n||_H = 1$ such that $||(i\beta_n I - A)U_n|| \to 0$, as $n \to \infty$, i.e.,

$$i\beta_n u_n - v_n \to 0 \quad \text{in} \quad H_0^1 \tag{3.2}$$

$$i\beta_n v_n - \alpha u_{n,xx} - \gamma(x) v_{n,xx} \to 0 \quad \text{in} \quad L^2. \tag{3.3}$$

Taking the inner product of $(i\beta_n I - A)U^n$ with U_n in H we obtain

$$((i\beta_n I - A)U_n, U_n) = i(\beta_n U_n, U_n) - (AU_n, U_n)$$

Taking the real part and using (2.1) we obtain

$$Re(\ (i\beta_n I - A)U_n, U_n\ ) \geq C \int_0^L |v_{n,x}|^2\ dx.$$

Noticing that (U_n) is bounded and $(i\beta_n I - A)U_n \to 0$ we obtain

$$C \int_0^L |v_{n,x}|^2\ dx \leq Re(\ (i\beta_n I - A)U_n, U_n\ ) \to 0 \quad \text{as} \quad n \to \infty, \tag{3.4}$$

and by Poincarè's Inequality

$$C \int_0^L |v_{n,x}|^2\ dx \geq \frac{C}{C_p} \int_0^L |v_n|^2\ dx\ . \tag{3.5}$$

It follows from (3.4) and (3.5) that $v_n \to 0 \quad \text{in} \quad L^2$. Then, from (3.2) it follows $u_n \to 0 \quad \text{in} \quad H_0^1$. This contradicts $||U_n||_H = 1$. Thus $\rho(A) \supset \{i\beta \mid \beta \in \mathbb{R}\}$ is proven.

Again, by a contradiction argument we will prove that

$$\limsup_{|\beta| \to \infty} ||(i\beta I - A)^{-1}|| < \infty. \tag{3.6}$$

If (3.6) is not true, then there exists a sequence $(V_n)_{n \in N}$ such that

$$\frac{||(\lambda_n I - A)^{-1} V_n||}{||V_n||} \geq n\ ,$$

where $\lambda_n = i\beta_n$. Hence

$$||(\lambda_n I - A)^{-1} V_n|| \geq n||V_n||\ .$$

Since $\lambda_n \in \rho(A)$ it follows that a unique sequence $(U_n) \in D(A)$ exists such that

$$\lambda_n U_n - AU_n = V_n, \quad ||U_n|| = 1 \ ,$$

that is, $U_n = (\lambda_n I - A)^{-1} V_n$ and

$$||U_n|| \geq n||\lambda_n U_n - AU_n||.$$

Now we denote $F_n = \lambda_n U_n - AU_n$ and observe that

$$||F_n|| \leq \frac{1}{n}||U_n||, \qquad ||U_n|| = 1.$$

Then

$$F_n \to 0 \text{ (strong)} \quad \text{in} \quad H \quad \text{as} \quad n \to \infty \ . \tag{3.7}$$

Now taking the inner product of F_n with U_n yields

$$\lambda_n(\ U_n, U_n \ ) - (\ AU_n, U_n \ ) = (\ F_n, U_n \ ), \qquad \lambda_n = i\beta_n. \tag{3.8}$$

Taking the real part of (3.8) we get from (2.1),

$$Re(\ F_n, U_n \ ) \geq \ C \int_0^L |v_{n,x}|^2 \ dx \ .$$

Noticing that (U_n) is bounded and that $F_n \to 0$ we obtain

$$C \int_0^L |v_{n,x}|^2 \ dx \leq \ Re \, (\ F_n, U_n \ ) \to 0 \ .$$

Using (3.5) we have

$$\int_0^L |v_n|^2 \ dx \to 0$$

from where we have

$$v_n \to 0 \text{ (strong)} \quad \text{in} \quad L^2 \ . \tag{3.9}$$

We put $F_n = (\ f_n, g_n \ ), \ U_n = (\ u_n, v_n \ )$. From $F_n = \lambda_n U_n - AU_n$ we have

$$\lambda_n u_n - v_n = f_n \quad \text{in} \quad H_0^1 \tag{3.10}$$

Now using (3.7), (3.9) and (3.10) we have

$$\lambda_n u_n \to 0.$$

Noticing that $\lambda_n \to \infty$, we conclude that $u_n \to 0$ (strong) in H_0^1. We again have a contradiction and the proof of the theorem is complete. ■

Acknowledgments

This paper was partially supported by FAPEMIG-MG. The authors would like to express gratitude to Professor Waldemar Donizete Bastos for the fruitful discussions concerning this paper and Professor Alexandre Nolasco de Carvalho for the careful analysis and suggestions presented.

C. A. R. da CUNHA; DINIZ, A. C. Estabilidade exponecial do semigrupo associado a um sistema termoviscoelástico com dissipação localmente distribuida. *Rev. Mat. Estat.*, São Paulo, v.24, n.2, p.07-15, 2006.

- RESUMO: Neste trabalhos estamos interessados em mostrar a estabilidade exponencial do semigrupo gerado pela equação de onda viscoelástica com dissipação localmente distribuída. Iremos utilizar o método desenvolvido por Z. Liu and S. Zheng. Este método é bem diferente de outros usados na literatura atual, tais como o tradicional método de Energia, o método de Kormonik e o método de Nakao.

- PALAVRAS-CHAVE: Decaimento exponencial; sistema dissipativo; sistema linear; dissipação localmente distribuída.

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Recebido em 15.06.2005.

Aprovado após revisão em 30.05.2006.