ON PARABOLIC EQUATIONS WITH LARGE DIFFUSION IN DUMBBELL DOMAINS

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▪ *ABSTRACT: In this paper we study the existence of patterns for parabolic equations in dumbbell-type domains. This is done in the case of large diffusivity and nonlinear boundary conditions.*

▪ *KEYWORDS: Dumbbell domains; parabolic equations; patterns; invariant manifolds.*

1 Introduction

The formation of nonconstant stable stationary solutions (*patterns*, for short) is of great interest in biological population growth, morphogenesis, neurobiology, theory of chemical reactors and many others fields. These problems are, in general, modeled by parabolic problems of the form

$$\begin{cases} u_t = d\,\Delta u + f(u), & (t,x) \in (0,\infty)\times\Omega, \\ \dfrac{\partial u}{\partial \nu} = 0, & (t,x) \in (0,\infty)\times\partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $d > 0$ is a diffusion coefficient, Δ is a Laplacian operator, $f : \mathbb{R} \to \mathbb{R}$ is a nonlinear reaction and $\partial/\partial\nu$ is the outer normal derivative on the boundary $\partial\Omega$ of Ω.

For $N = 1$ and Ω being an interval, it was shown that any stable stationary solution of (1.1) is constant (see Chaffe (1975)). Latter, Matano (1979) obtained

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the same result when $\Omega \subset \mathbb{R}^N$ is a *convex* domain and for any nonlinearity f . Also Matano (1979) shows that there exists a *non-convex* domain $\Omega \subset \mathbb{R}^2$ for which (1.1) has patterns with certain additional conditions in the function f. The case of systems was considered in Kishimoto and Weinberger (1985).

This result indicates that the stability of spatially nonconstant stable equilibria depends on the form of the domain. The idea is that the term $d\Delta u$ promotes homogenization of solutions as time evolves and, when the domain is convex, any gradient of concentration can be dissipated taking the shortest path. This effect rules out stable nonconstant equilibria for convex domains. Taking away convexity, we may allow patterns to appear.

Morita (1990) studied problem (1.1) when d is a fixed large parameter and Ω is a non-convex domain in the form of a dumbbell (two disconnected domains joined by a thin channel). He shows the existence of a finite dimensional exponentially attracting invariant manifold and the reduced form of the differential equations in the invariant manifold.

In this paper, we study the formation of patterns for the case when the reaction also happens at the boundary of the domain. More specifically, we consider the parabolic problems with nonlinear Neumann boundary conditions with two parameters $\epsilon > 0$ and $\theta > 0$

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{\epsilon^\theta}\Delta u + f(u) & \text{in} \quad \Omega_\epsilon \\ \dfrac{1}{\epsilon^\theta}\dfrac{\partial u}{\partial \nu} = g(u) & \text{on} \quad \partial\Omega_\epsilon. \end{cases} \tag{1.2}$$

Here $\Omega_\epsilon \subset \mathbb{R}^N$ is a bounded smooth domain consisting of two fixed disconnected parts and a channel (ϵ-dependent) connecting them (see FIGURE-1), $\frac{\partial}{\partial \nu}$ is the outer normal derivative on the boundary $\partial\Omega_\epsilon$ of Ω_ϵ, and f, g are nonlinear functions satisfying certain growth and dissipativeness conditions that will be specified later.

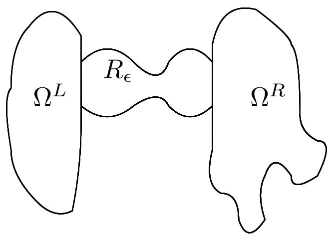

Figure 1 - Dumbbell type domain.

Our goal in this paper is to show that, for suitably small ϵ, the dynamics of the equation (1.2) is given by an ordinary differential equation (or a system of ODE's weakly coupled). The results here extend the results in Carvalho and Lozada-Cruz (2006) for we analyze the different situations that arise when parameter θ varies.

Let us now introduce the assumptions on the nonlinearities f and g. Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are twice continuously differentiable functions satisfying

$$\limsup_{|u|\to\infty} \frac{f(u)}{u} < 0 \quad \text{and} \quad \limsup_{|u|\to\infty} \frac{g(u)}{u} < 0.$$

In addition, assume some growth assumptions to ensure local well posedness of (1.2) (see Arrieta, Carvalho, and Rodriguez-Bernal (2000)). Under these assumptions, problem (1.2) has a global attractor $\mathcal{A}_\epsilon$, $0 \leqslant \epsilon \leqslant \epsilon_0$ which is compact, invariant, attracts each bounded subset of $H^1(\Omega_\epsilon)$ and $\sup\limits_{0\leqslant\epsilon\leqslant\epsilon_0} \sup\limits_{u\in\mathcal{A}_\epsilon} \|u\|_{L^\infty(\Omega_\epsilon)} < \infty$ (see Arrieta, Carvalho, and Rodriguez-Bernal (2000)). This enables us to cut f and g in such a way that attractors $\mathcal{A}_\epsilon$ remain the same and in such a way that f and g together with their derivatives up to second order are bounded. Hereafter we assume (without loss of generality) that f and g are bounded functions with bounded derivatives up to second order.

Next we specify domain $\Omega_\epsilon \subset \mathbb{R}^N$ ($N \geqslant 2$). It has a fixed part Ω and a parameter-dependent part R_ϵ, that is, $\Omega_\epsilon = \Omega \cup R_\epsilon$, where $\Omega = \Omega^{\mathrm{L}} \cup \Omega^{\mathrm{R}}$. We assume that Ω^{L}, Ω^{R} and R_ϵ satisfy the following conditions:

(**I**) Ω^{L}, Ω^{R} are bounded smooth domains in $\mathbb{R}^N$ with disjoint closures,

(**II**) There is an orthogonal system of coordinates $x = (x_1, x_2, \cdots, x_N) = (x_1, x') \in \mathbb{R}^N$ such that the following conditions hold for some positive constant $\epsilon^* > 0$,

$$\overline{\Omega}^{\mathrm{L}} \cap \{(x_1, x') \in \mathbb{R}^N : x_1 \geqslant 0,\ |x'| < \epsilon^*\} = \left\{(0, x') \in \mathbb{R}^N : |x'| < \epsilon^*\right\}$$
$$\overline{\Omega}^{\mathrm{R}} \cap \{(x_1, x') \in \mathbb{R}^N : x_1 \leqslant 1,\ |x'| < \epsilon^*\} = \left\{(1, x') \in \mathbb{R}^N : |x'| < \epsilon^*\right\},$$

(**III**) $R_\epsilon = \left\{(x_1, x') \in \mathbb{R}^N : 0 < x_1 < 1,\ |x'| < \epsilon\, h(x_1)\right\}$ where $h \in \mathcal{C}^0([0,1]) \cap \mathcal{C}^1((0,1))$ and $h(x_1) > 0$ for all $x_1 \in [0,1]$.

Without loss of generality, we assume that $|\Omega| = 1$ throughout the paper.

If we denote by $\{\lambda_n^\epsilon\}_{n=1}^\infty$ and $\{\varphi_n^\epsilon\}_{n=1}^\infty$ the set of eigenvalues and corresponding orthonormalized eigenfunctions for the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda^\epsilon u, & \Omega_\epsilon, \\ \dfrac{\partial u}{\partial \nu} = 0, & \partial\Omega_\epsilon, \end{cases} \tag{1.3}$$

then we have the following result.

Theorem 1.1. The following holds:

$$\lambda_1^\epsilon = 0 \text{ and } \varphi_1^\epsilon = |\Omega_\epsilon|^{-1/2}, \ \forall \epsilon > 0$$

$$\lim_{\epsilon\to 0} \frac{\lambda_2^\epsilon}{\epsilon^{N-1}} = \mathrm{C}_0, \quad \varphi_2^\epsilon \overset{\epsilon\to 0}{\longrightarrow} \begin{cases} c_1^0, & \text{in } \Omega^{\mathrm{L}}, \\ \\ c_2^0, & \text{in } \Omega^{\mathrm{R}}, \end{cases} \quad \text{in } L^2(\Omega) \text{ and } L^2(\partial\Omega)$$

$$\sup_{\epsilon>0} \|\varphi_2^\epsilon\|_{L^\infty(\Omega_\epsilon)} < \infty,$$

$$\liminf_{\epsilon\to 0} \lambda_3^\epsilon > 0$$

where $\mathrm{C}_0 = \sigma_{N-1}\left(\frac{1}{|\Omega^{\mathrm{L}}|} + \frac{1}{|\Omega^{\mathrm{R}}|}\right)\left\{\int_0^1 \frac{dx_1}{h^{N-1}(x_1)}\right\}^{-1}$ and σ_{N-1} is the Lebesgue measure of the unit ball in $\mathbb{R}^{N-1}$, $c_1^0 := -(|\Omega^{\mathrm{R}}|/|\Omega^{\mathrm{L}}|)^{1/2}$, and $c_2^0 := (|\Omega^{\mathrm{L}}|/|\Omega^{\mathrm{R}}|)^{1/2}$.

A proof of this result (with stronger convergence properties for the eigenfunctions) is given in Carvalho and Lozada-Cruz (2006), and it is sketched in Appendix A for completeness.

This paper is organized as follows. In Section 2, we state the main results of the paper. In section 3, we give an example for which we can obtain stable nonconstant equilibria from the boundary reaction. At the end of the paper, we include one appendix where we prove Theorem 1.1 (Appendix A).

2 Main results

If u is a solution of problem (1.2), we have that u can be written as $u(t,x) = u_1(t)\varphi_1^\epsilon(x) + u_2(t)\varphi_2^\epsilon(x) + w(t,x)$, where

$$u_1 = \int_{\Omega_\epsilon} u\varphi_1^\epsilon, \ \ u_2 = \int_{\Omega_\epsilon} u\varphi_2^\epsilon, \ \ w = u - u_1\varphi_1^\epsilon - u_2\varphi_2^\epsilon.$$

This decomposition induces the following decomposition of (1.2)

$$\begin{aligned} \dot{u}_1 &= \int_{\Omega_\epsilon} f(u)\varphi_1^\epsilon + \int_{\partial\Omega_\epsilon} g(u)\varphi_1^\epsilon, \\ \dot{u}_2 &= -\frac{\lambda_2^\epsilon}{\epsilon^\theta}u_2 + \int_{\Omega_\epsilon} f(u)\varphi_2^\epsilon + \int_{\partial\Omega_\epsilon} g(u)\varphi_2^\epsilon, \\ w_t &= \frac{1}{\epsilon^\theta}\Delta w + f(u) - \left[\int_{\Omega_\epsilon} f(u)\varphi_1^\epsilon + \int_{\partial\Omega_\epsilon} g(u)\varphi_1^\epsilon\right]\varphi_1^\epsilon - \\ &\quad \left[\int_{\Omega_\epsilon} f(u)\varphi_2^\epsilon + \int_{\partial\Omega_\epsilon} g(u)\varphi_2^\epsilon\right]\varphi_2^\epsilon, \\ \frac{1}{\epsilon^\theta}\frac{\partial w}{\partial\nu} &= g(u_1\varphi_1^\epsilon + u_2\varphi_2^\epsilon + w). \end{aligned} \tag{2.4}$$

Now defining

$$G_1^\epsilon(u_1,u_2,w) = \int_{\Omega_\epsilon} f(u)\varphi_1^\epsilon + \int_{\partial\Omega_\epsilon} g(u)\varphi_1^\epsilon, \tag{2.5}$$

$$G_2^\epsilon(u_1,u_2,w) = \int_{\Omega_\epsilon} f(u)\varphi_2^\epsilon + \int_{\partial\Omega_\epsilon} g(u)\varphi_2^\epsilon, \tag{2.6}$$

we obtain the following system for u_1, u_2 and w

$$\begin{cases} \dot{u}_1 = G_1^\epsilon(u_1,u_2,w), \\ \dot{u}_2 = -\dfrac{\lambda_2^\epsilon}{\epsilon^\theta}u_2 + G_2^\epsilon(u_1,u_2,w), \\ w_t = \dfrac{1}{\epsilon^\theta}\Delta w + f(u) - G_1^\epsilon(u_1,u_2,w)\varphi_1^\epsilon - G_2^\epsilon(u_1,u_2,w)\varphi_2^\epsilon, \\ \dfrac{1}{\epsilon^\theta}\dfrac{\partial w}{\partial \nu} = g(u_1\varphi_1^\epsilon + u_2\varphi_2^\epsilon + w). \end{cases} \tag{2.7}$$

Define $\gamma(\theta) = \lim_{\epsilon\to 0} \dfrac{\lambda_2^\epsilon}{\epsilon^\theta}$. Then, by the Theorem 1.1 we have

$$\gamma(\theta) = \begin{cases} 0, & \text{if} \quad 0 < \theta < N-1, \\ \mathrm{C}_0, & \text{if} \quad \theta = N-1 \\ \infty, & \text{if} \quad \theta > N-1. \end{cases} \tag{2.8}$$

In what follows, we give a heuristic argument to find the limiting ode that should contain the asymptotic behavior of (2.7). By Theorem 1.1, the third eigenvalue $\frac{\lambda_3^\epsilon}{\epsilon^\theta}$ (of $-\frac{1}{\epsilon^\theta}\Delta$ with Neumann boundary condition) goes to infinity as $\epsilon \to 0^+$, we guess that ω does not play any role in the asymptotic behavior of (1.2); then, for $0 < \theta \leqslant N-1$,

$$\begin{cases} \dot{u}_1 \sim \int_{\Omega_\epsilon} f(u_1\varphi_1^\epsilon + u_2\varphi_2^\epsilon)\varphi_1^\epsilon + \int_{\partial\Omega_\epsilon} g(u_1\varphi_1^\epsilon + u_2\varphi_2^\epsilon)\varphi_1^\epsilon \\ \\ \dot{u}_2 \sim -\dfrac{\lambda_2^\epsilon}{\epsilon^\theta}\, u_2 + \int_{\Omega_\epsilon} f(u_1\varphi_1^\epsilon + u_2\varphi_2^\epsilon)\varphi_2^\epsilon + \int_{\partial\Omega_\epsilon} g(u_1\varphi_1^\epsilon + u_2\varphi_2^\epsilon)\varphi_2^\epsilon. \end{cases} \tag{2.9}$$

According to Theorem 1.1, we should have the limit system given by

$$\begin{cases} \dot{u}_1 = |\Omega^{\mathrm{L}}| f(u_1 + c_1^0 u_2) + |\Omega^{\mathrm{R}}| f(u_1 + c_2^0 u_2) \\ \qquad + |\partial\Omega^{\mathrm{L}}| g(u_1 + c_1^0 u_2) + |\partial\Omega^{\mathrm{R}}| g(u_1 + c_2^0 u_2) =: G_1^0(u_1,u_2), \\ \dot{u}_2 = -\gamma(\theta)\, u_2 + |\Omega^L| c_1^0 f(u_1 + c_1^0 u_2) + |\Omega^{\mathrm{R}}| c_2^0 f(u_1 + c_2^0 u_2) \\ \qquad + |\partial\Omega^{\mathrm{L}}| c_1^0 g(u_1 + c_1^0 u_2) + |\partial\Omega^{\mathrm{R}}| c_2^0 g(u_1 + c_2^0 u_2) =: G_2^0(u_1,u_2), \end{cases} \tag{2.10}$$

where c_1^0, c_2^0 and C_0 given in Theorem 1.1.

Variables u_1 and u_2 may not be the best choice of variables to study this problem. A better choice of variables is probably that which reflects the averages over Ω^{L} and Ω^{R}. To relate u_1 and u_2 with these averages we consider

$$v_1 = |\Omega^{\rm L}|^{-1} \int_{\Omega^{\rm L}} u(t,x)dx \quad \text{and} \quad v_2 = |\Omega^{\rm R}|^{-1} \int_{\Omega^{\rm R}} u(t,x)dx.$$

Thus $u_1 = |\Omega^{\rm L}|v_1 + |\Omega^{\rm R}|v_2,\ u_2 = -\big(|\Omega^{\rm L}||\Omega^{\rm R}|\big)^{1/2}(v_1 - v_2)$ and

$$\begin{cases} v_1 = u_1 + c_1^0 u_2 \\ v_2 = u_1 + c_2^0 u_2. \end{cases} \tag{2.11}$$

Using (2.11) the system (2.10), for $0 < \theta \leqslant N-1$ becomes

$$\begin{cases} \dot v_1 = -\gamma(\theta)\, |\Omega^{\rm R}|(v_1 - v_2) + f(v_1) + \dfrac{|\partial\Omega^{\rm L}|}{|\Omega^{\rm L}|} g(v_1) \\ \dot v_2 = \gamma(\theta)\, |\Omega^{\rm L}|(v_1 - v_2) + f(v_2) + \dfrac{|\partial\Omega^{\rm R}|}{|\Omega^{\rm R}|} g(v_2). \end{cases} \tag{2.12}$$

Case $\theta > N-1$ is a lot simpler since in this case $\frac{\lambda_2^\epsilon}{\epsilon^\theta} \to \infty$ as $\epsilon \to 0$. Hence, u_2 and ω should play no role in the asymptotic dynamics. That leads to

$$\dot u_1 \sim \int_{\Omega_\epsilon} f(u_1\varphi_1^\epsilon)\varphi_1^\epsilon + \int_{\partial\Omega_\epsilon} g(u_1\varphi_1^\epsilon)\varphi_1^\epsilon. \tag{2.13}$$

and the limiting equation should be

$$\dot u_1 = |\Omega| f(u_1) + |\partial\Omega| g(u_1). \tag{2.14}$$

Now our aim is to show that the dynamics of (1.2) can be described by the dynamics of (2.12) if $0 < \theta \leq N-1$ or by (2.14) if $\theta > N-1$. With this in hand, in case $0 < \theta < N-1$, we will try to produce asymptotically stable equilibrium solutions (v_1, v_2) for (2.12) with $v_1 \neq v_2$ and they should correspond to asymptotically stable equilibrium solutions to (1.2) which are nonconstant for v_1 reflects the average in $\Omega^{\rm L}$ and v_2 reflects the average in $\Omega^{\rm R}$.

To justify the intuitive procedure described above we use the theory of invariant manifolds. Before we proceed, we need to introduce some terminology. For $\epsilon > 0$, let $X_\epsilon = L^2(\Omega_\epsilon)$ and $L_\epsilon : D(L_\epsilon) \subset X_\epsilon \to X_\epsilon$ be the operator defined by

$$D(L_\epsilon) = \{u \in H^2(\Omega_\epsilon) :\ \frac{\partial u}{\partial n} = 0\}$$
$$L_\epsilon u = \frac{1}{\epsilon^\theta}\,\Delta u.$$

It is well known that L_ϵ is an unbounded, self adjoint, non-positive definite operator which has compact resolvent. It follows that $-L_\epsilon$ is a sectorial operator and, for $\zeta > 0$ fixed, we can define the fractional powers $(-L_\epsilon + \zeta I)^\alpha$ and corresponding fractional power spaces X_ϵ^α ($X_\epsilon^\alpha = D((-L_\epsilon + \zeta I)^\alpha)$ endowed with the graph norm), $\alpha > 0$. X_ϵ^α is a Hilbert space with the inner product

$\langle\varphi,\psi\rangle_\alpha = \int_{\Omega_\epsilon}(-L_\epsilon+\zeta I)^\alpha\varphi\ (-L_\epsilon+\zeta I)^\alpha\psi$. Then, $X_\epsilon^{\frac{1}{2}} = H^1(\Omega_\epsilon)$ (with norm $\|\psi\|^2_{X_\epsilon^{\frac{1}{2}}} = \frac{1}{\epsilon^\theta}\|\nabla\psi\|^2_{L^2(\Omega_\epsilon)} + \zeta\|\psi\|^2_{L^2(\Omega_\epsilon)}$) and $X_\epsilon^1 = D(L_\epsilon)$ with the graph norm.

Also, let $V = \text{span}[\varphi_1^\epsilon, \varphi_2^\epsilon]$, $V^\perp = \{\varphi \in L^2(\Omega_\epsilon) : \int_{\Omega_\epsilon}\varphi\psi dx = 0,\ \text{for all } \psi \in V\}$ and $V_\alpha^\perp = V^\perp \cap X_\epsilon^\alpha = \{\varphi \in X_\epsilon^\alpha : \langle\varphi,\psi\rangle_\alpha = 0,\ \text{for all } \psi \in V\}$. Define $A_\epsilon : D(A_\epsilon) \subset V^\perp \to V^\perp$ by $D(A_\epsilon) = V_1^\perp$ and $A_\epsilon\omega = L_\epsilon\omega$ for all $\omega \in D(A_\epsilon)$; that is, the restriction of L_ϵ to $V^\perp$. We also denote by A_ϵ its realization in $X_\epsilon^{\frac{1}{2}}$. Also define $B_\epsilon = \text{diag}\Big(0, -\frac{\lambda_2^\epsilon}{\epsilon^\theta}\Big)$.

Making $y = (u_1, u_2)$ we have that (2.7) can be rewritten as

$$\begin{cases} \dot\omega &= A_\epsilon\omega + F_\epsilon(y,\omega), \\ \dot y &= B_\epsilon y + G_\epsilon(y,\omega), \end{cases} \tag{2.15}$$

where $G_\epsilon(y,\omega) = (G_1^\epsilon(u_1,u_2,\omega),\ G_2^\epsilon(u_1,u_2,\omega))$ and $F_\epsilon(u_1,u_2,\omega) = f(u) - G_1^\epsilon(u_1,u_2,\omega)\varphi_1^\epsilon - G_2^\epsilon(u_1,u_2,\omega)\varphi_2^\epsilon$.

The following result shows the existence of invariant manifold.

Theorem 2.1. There is a continuously differentiable map $\sigma_\epsilon : \mathbb{R}^2 \to V_{\frac{1}{2}}^\perp$ such that $S_\epsilon = \Big\{((u_1,u_2),\omega) \in \mathbb{R}^2 \oplus V_{\frac{1}{2}}^\perp : \omega = \sigma_\epsilon(u_1,u_2)\Big\}$ is an exponentially attracting invariant manifold for (2.7). The flow on S_ϵ is given by $u(t,x) = u_1(t)\varphi_1^\epsilon(x) + u_2(t)\varphi_2^\epsilon(x) + \sigma_\epsilon(u_1(t),u_2(t))$, where (u_1,u_2) is the solution of

$$\begin{cases} \dot u_1 &= G_1^\epsilon(u_1,u_2,\sigma_\epsilon(u_1,u_2)) \\ \dot u_2 &= -\dfrac{\lambda_2^\epsilon}{\epsilon^\theta}\ u_2 + G_2^\epsilon(u_1,u_2,\sigma_\epsilon(u_1,u_2)). \end{cases} \tag{2.16}$$

Furthermore, $\sigma_\epsilon \to 0$ in $\mathcal{C}^1(\mathbb{R}^2, V_{\frac{1}{2}}^\perp)$, when $\epsilon \to 0$.

Proof. The proof is adapted from (Carvalho and Lozada-Cruz (2006), Theorem 2.2). A sketch of it is added here for completeness. We can divide the proof in three cases:

Case (i). $0 < \theta < N-1$. According with Theorem 1.1, $\frac{\lambda_1^\epsilon}{\epsilon^\theta} \equiv 0$, $\frac{\lambda_2^\epsilon}{\epsilon^\theta} \overset{\epsilon\to 0}{\longrightarrow} 0$ and $\frac{\lambda_3^\epsilon}{\epsilon^\theta} \overset{\epsilon\to 0}{\longrightarrow} \infty$. As a consequence of the large gap existing between $\frac{\lambda_2^\epsilon}{\epsilon^\theta}$ and $\frac{\lambda_3^\epsilon}{\epsilon^\theta}$, proceeding as in Carvalho and Lozada-Cruz (2006) and Lozada-Cruz (2004), we obtain the existence of an invariant manifold S_1 for (2.7) which it is a graph of a function $\sigma_\epsilon^1 : \mathbb{R}^2 \to V_{\frac{1}{2}}^\perp$ with $\sigma_\epsilon^1 \to 0$ in $\mathcal{C}^1(\mathbb{R}^2, V_{\frac{1}{2}}^\perp)$, when $\epsilon \to 0$. Using the convergence of eigenvalues and eigenfunctions (Theorem 1.1) we have

$$\begin{aligned} G_1^\epsilon(u_1,u_2,\sigma_\epsilon^1(u_1,u_2)) &\to G_1^0(u_1,u_2) \\ G_2^\epsilon(u_1,u_2,\sigma_\epsilon^1(u_1,u_2)) &\to G_2^0(u_1,u_2) \end{aligned}$$

as $\epsilon \to 0$ and using the change of variables (2.11), we have a decoupled ($\gamma(\theta) = 0$) system of ODE's in the variables v_1 and v_2.

Case (ii). $\theta = N-1$. This is exactly the case considered in Carvalho and Lozada-Cruz (2006) and Lozada-Cruz (2004). According with Theorem 1.1, $\frac{\lambda_1^\epsilon}{\epsilon^{N-1}} \equiv 0$, $\frac{\lambda_2^\epsilon}{\epsilon^{N-1}} \stackrel{\epsilon\to 0}{\longrightarrow} \mathrm{C}_0$ and $\frac{\lambda_3^\epsilon}{\epsilon^{N-1}} \stackrel{\epsilon\to 0}{\longrightarrow} \infty$. As a consequence of the large gap existing between $\frac{\lambda_2^\epsilon}{\epsilon^{N-1}}$ and $\frac{\lambda_3^\epsilon}{\epsilon^{N-1}}$, we obtain the existence of an invariant manifold S_2 for (2.7) which it is graph of a function $\sigma_\epsilon^2 : \mathbb{R}^2 \to V_{\frac{1}{2}}^\perp$ with $\sigma_\epsilon^2 \to 0$ in $\mathcal{C}^1(\mathbb{R}^2, V_{\frac{1}{2}}^\perp)$, when $\epsilon \to 0$. Using the convergence of eigenvalues and eigenfunctions (Theorem 1.1) we have

$$G_1^\epsilon(u_1,u_2,\sigma_\epsilon^2(u_1,u_2)) \to G_1^0(u_1,u_2)$$
$$G_2^\epsilon(u_1,u_2,\sigma_\epsilon^2(u_1,u_2)) \to G_2^0(u_1,u_2)$$

when $\epsilon \to 0$ and using the change of variables (2.11) we have a system of ODE's weakly coupled ($\gamma(N-1) = \mathrm{C}_0$) in the variables v_1 and v_2.

Case (iii). $\theta > N-1$. According to Theorem 1.1, $\frac{\lambda_1^\epsilon}{\epsilon^\theta} \equiv 0$, $\frac{\lambda_2^\epsilon}{\epsilon^\theta} \stackrel{\epsilon\to 0}{\longrightarrow} \infty$ and $\lambda_2^\epsilon - \lambda_3^\epsilon \stackrel{\epsilon\to 0}{\longrightarrow} \infty$. Since the gap between the second and third eigenvalue goes to infinity we still have an invariant manifold S_3 for (2.7) which is a graph of a function $\sigma_\epsilon^3 : \mathbb{R}^2 \to V_{\frac{1}{2}}^\perp$ with $\sigma_\epsilon^3 \to 0$ in $\mathcal{C}^1(\mathbb{R}^2, V_{\frac{1}{2}}^\perp)$, when $\epsilon \to 0$. However, since $\dfrac{\lambda_2^\epsilon}{\epsilon^\theta} \to +\infty$ when $\epsilon \to 0$ we also have an invariant manifold S_4 which is the graph of a function $\sigma_\epsilon^4 : \mathbb{R} \to [\varphi_1^\epsilon]^\perp$ such that

The flow on S_4 is given by $u(t,x) = u_1(t)\varphi_1^\epsilon(x) + \sigma_\epsilon^4(u_1(t))$, where u_1 is the solution of

$$\dot{u}_1 = G_1^\epsilon(u_1, \sigma_\epsilon^4(u_1)). \tag{2.17}$$

Furthermore, $\sigma_\epsilon^4 \to 0$ in $\mathcal{C}^1(\mathbb{R}, [\varphi_1^\epsilon]^\perp)$, when $\epsilon \to 0$. Using this and the convergence of eigenvalues and eigenfunctions, we have that

$$G_1^\epsilon(u_1, \sigma_\epsilon^4(u_1)) \to G_1^0(u_1, 0). \quad \blacksquare$$

The following result tell us that the dynamics of systems (2.12) and (2.16) are equivalent (if (2.12) is structurally stable), thus in particular stable equilibria for (2.12) correspond to stable nonconstant equilibria of (2.16).

Theorem 2.2. Assume that the system (2.12) is structurally stable. Then, for small enough ϵ, the flow on the invariant manifold given by (2.16) is topological equivalent to the flow (2.12).

Proof. We only consider the case $\theta = N-1$, the remaining cases are similar. For suitably small ϵ, the flow in the invariant manifold is given by $v(t,x) = u_1(t)\varphi_1^\epsilon(x) + u_2(t)\varphi_2^\epsilon(x) + \sigma_\epsilon(u_1(t), u_2(t))(x)$, where $(u_1(t), u_2(t))$ is a solution of

$$\begin{cases} \dot{u}_1 &= G_0^\epsilon(u_1, u_2, \sigma_\epsilon(u_1,u_2)) = X_0(\epsilon) \\ \dot{u}_2 &= -\frac{\lambda_2^\epsilon}{\epsilon^{N-1}} u_2 + G_1^\epsilon(u_1,u_2,\sigma_\epsilon(u_1,u_2)) = X_1(\epsilon). \end{cases} \tag{2.18}$$

To obtain that the flow on the attractor of (2.10) (or equivalently (2.12)) is topologically equivalent to the flow on the attractor of (2.18), we only need to prove that the vector fields

$$X_\epsilon(u_1,u_2) = (X_0(\epsilon), X_1(\epsilon)) \quad \text{and} \quad X_0(u_1,u_2) = (X_0(0), X_1(0)),$$

where

$$\begin{aligned}
X_0(\epsilon) =& \int_{\Omega_\epsilon} f(u_1\varphi_1^\epsilon + u_2\varphi_2^\epsilon + \sigma_\epsilon(u_1,u_2))\varphi_1^\epsilon + \int_{\partial\Omega_\epsilon} g(u_1\varphi_1^\epsilon + u_2\varphi_2^\epsilon + \sigma_\epsilon(u_1,u_2))\varphi_1^\epsilon \\
X_1(\epsilon) =& -\frac{\lambda_2^\epsilon}{\epsilon^{N-1}}\, u_2 + \int_{\Omega_\epsilon} f(u_1\varphi_1^\epsilon + u_2\varphi_2^\epsilon + \sigma_\epsilon(u_1,u_2))\varphi_2^\epsilon \\
&+ \int_{\partial\Omega_\epsilon} g(u_1\varphi_1^\epsilon + u_2\varphi_2^\epsilon + \sigma_\epsilon(u_1,u_2))\varphi_2^\epsilon \\
X_0(0) =& \int_\Omega f(u_1 + u_2\phi_2) + \int_{\partial\Omega} g(u_1 + u_2\phi_2) \\
X_1(0) =& -\mathrm{C}_0 u_2 + \int_\Omega f(u_1 + u_2\phi_2)\phi_2 + \int_{\partial\Omega} g(u_1 + u_2\phi_2)\phi_2
\end{aligned}$$

are $\mathcal{C}^1$ close. This follows easily from the fact that σ_ϵ approaches 0 in the $\mathcal{C}^1(\mathbb{R}^2, V_{\frac{1}{2}}^{\perp})$ topology and the asymptotic properties of the eigenvalues and eigenfunctions of $-\frac{1}{\epsilon^{N-1}}\Delta$ as $\epsilon \to 0$. Just to give an idea of the techniques involved, let us prove that $\partial_{u_1} X_1(\epsilon)$ converges to $\partial_{u_1} X_1(0)$ as $\epsilon \to 0$. Note that

$$\begin{aligned}
&\partial_{u_1} X_1(\epsilon)(u_1,u_2) - \partial_{u_1} X_1(0)(u_1,u_2) \\
&\quad = \int_{\Omega_\epsilon} f'(u_1\varphi_1^\epsilon + u_2\varphi_2^\epsilon + \sigma_\epsilon(u_1,u_2))\varphi_2^\epsilon\varphi_1^\epsilon - \int_\Omega f'(u_1 + u_2\phi_2)\phi_2 \\
&\quad + \int_{\partial\Omega_\epsilon} g'(u_1\varphi_1^\epsilon + u_2\varphi_2^\epsilon + \sigma_\epsilon(u_1,u_2))\varphi_2^\epsilon\varphi_1^\epsilon - \int_{\partial\Omega} g'(u_1 + u_2\phi_2)\phi_2 \\
&\quad + \int_{\Omega_\epsilon} f'(u_1\varphi_1^\epsilon + u_2\varphi_2^\epsilon + \sigma_\epsilon(u_1,u_2))\, \varphi_2^\epsilon\, \partial_{u_1}\sigma_\epsilon(u_1,u_2). \\
&\quad + \int_{\partial\Omega_\epsilon} g'(u_1\varphi_1^\epsilon + u_2\varphi_2^\epsilon + \sigma_\epsilon(u_1,u_2))\, \varphi_2^\epsilon\, \partial_{u_1}\sigma_\epsilon(u_1,u_2).
\end{aligned}$$

All but the last two lines in the above expression go to zero, uniformly in bounded subsets of $\mathbb{R}^2$, because of the convergence properties of $\lambda_1^\epsilon, \lambda_2^\epsilon$, φ_1^ϵ and φ_2^ϵ and because $\|\sigma_\epsilon(u_1,u_2)\|_{V_{\frac{1}{2}}^{\perp}} \to 0$. For the last two lines we only need to observe that $\|\partial_{u_1}\sigma_\epsilon(u_1,u_2)\|_{V_{\frac{1}{2}}^{\perp}} \to 0$ and use the Uniform Trace Theorem (Carvalho and Lozada-Cruz (2006), Theorem 6) to conclude that $\|\partial_{u_1}\sigma_\epsilon(u_1,u_2)\|_{L^2(\partial\Omega_\epsilon)} \overset{\epsilon\to 0}{\longrightarrow} 0$. ■

Remark 2.3. Making $|\Omega^{\mathrm{L}}| = |\Omega^{\mathrm{R}}|$, $|\partial\Omega^{\mathrm{L}}| = |\partial\Omega^{\mathrm{R}}|$, $r := \mathrm{C}_0|\Omega^{\mathrm{L}}| = \mathrm{C}_0\,|\Omega^{\mathrm{R}}|$ and $f(s) + \frac{|\partial\Omega^{\mathrm{L}}|}{|\Omega^{\mathrm{L}}|} g(s) = f(s) + \frac{|\partial\Omega^{\mathrm{R}}|}{|\Omega^{\mathrm{R}}|} g(s) := s - s^3$ we can verify the conditions of (

Andronov, Vitt, and Khaĭkin (1987), Theorem V, page 395) for $r \neq \frac{1}{2}$ and $r \neq \frac{1}{3}$ (see Carvalho (1995), page 400). Hence, in this case, (2.12) is structurally stable. For the general case, see Fusco and Oliva (1988) for conditions ensuring that (2.12) is structurally stable.

3 Patterns in the parabolic equation

Equation (1.2) was studied by Fang (1990) when $N = 2$ with homogeneous Neumann boundary conditions ($g = 0$). In this paper, we consider $N > 2$ with nonlinear boundary conditions.

Using (2.8) we have that limit equations for (2.12) as $\epsilon \to 0$ are
(**i**) $0 < \theta < N - 1$

$$\begin{cases} \dot{v}_1 = f(v_1) + \dfrac{|\partial\Omega^{\mathrm{L}}|}{|\Omega^{\mathrm{L}}|} g(v_1) = h_1(v_1) \\ \dot{v}_2 = f(v_2) + \dfrac{|\partial\Omega^{\mathrm{R}}|}{|\Omega^{\mathrm{R}}|} g(v_2) = h_2(v_2), \end{cases} \tag{3.19}$$

(**ii**) $\theta = N - 1$

$$\begin{cases} \dot{v}_1 = -\mathrm{C}_0\, |\Omega^{\mathrm{R}}|(v_1 - v_2) + f(v_1) + \dfrac{|\partial\Omega^{\mathrm{L}}|}{|\Omega^{\mathrm{L}}|} g(v_1) \\ \dot{v}_2 = \mathrm{C}_0\, |\Omega^{\mathrm{L}}|(v_1 - v_2) + f(v_2) + \dfrac{|\partial\Omega^{\mathrm{R}}|}{|\Omega^{\mathrm{R}}|} g(v_2), \end{cases} \tag{3.20}$$

(**iii**) $\theta > N - 1$, $u = v_1 = v_2$,

$$\dot{u} = f(u) + |\partial\Omega|\, g(u). \tag{3.21}$$

Observe that:

• In the case (**i**) there are patterns for (1.2), whenever the stable equilibria of h_1 and h_2 be distinct.

For better understanding we put $g(u) = u - u^3$ and $f(u) = 0$. Thus the system (3.19) has nine equilibrium points (see FIGURE-2).

$$\begin{aligned} \mathrm{P}_1 &= (0,0), \quad \mathrm{P}_2 = (0,1), \quad \mathrm{P}_3 = (0,-1), \\ \mathrm{P}_4 &= (1,0), \quad \mathrm{P}_5 = (1,1), \quad \mathrm{P}_6 = (1,-1), \\ \mathrm{P}_7 &= (-1,0), \quad \mathrm{P}_8 = (-1,1), \quad \mathrm{P}_9 = (-1,-1). \end{aligned}$$

All of these equilibrium points are hyperbolic. Equilibria P_5, P_6, P_8, P_9 are stable. We have seen that the dynamics of (3.19) is equivalent to that of (1.2), the equilibrium points of form $P = (v_1, v_2)$ with $v_1 \neq v_2$ correspond to stable nonconstant equilibria (patterns) for (1.2). Thus, we have two patterns for the system (1.2) (see FIGURE-2).

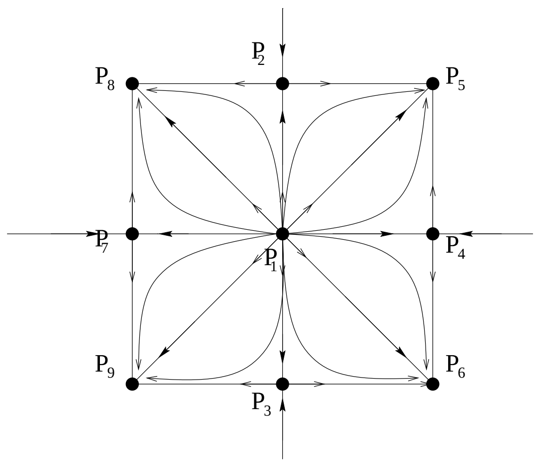

Figure 2 - Approximate phase portrait for $g(u) = u - u^3$, $f = 0$.

• In case (**ii**), we will also have the existence of patterns for (1.2) as long as C_0 is suitably small (for more details see Carvalho and Lozada-Cruz (2006) and Lozada-Cruz (2004)).
• In case (**iii**), we are not able to guarantee the existence of patterns for (1.2) with the technique used here. In this case all of the equilibria (stable and unstable) are approximately constant.

A Convergence of Eigenvalues and Eigenfunctions

In this appendix we show Theorem 1.1 concerning eigenvalues λ_n^ϵ arranged in increasing order (counting multiplicity) and a complete system of orthonormalized eigenfunctions φ_n^ϵ associated with problem (1.3). Let $\phi_1 = |\Omega|^{-1/2} = 1$ and $\phi_2 = c_1^0 \chi_{\Omega^{\mathrm{L}}} + c_2^0 \chi_{\Omega^{\mathrm{R}}}$.

For the results in this appendix we follow the ideas in Arrieta (1991) and Arrieta (1995).

The exact rate of convergence of λ_2^ϵ to 0 is given by the following proposition.

Proposition A.1.

$$\lim_{\epsilon \to 0} \frac{\lambda_2^\epsilon}{\epsilon^{N-1}} = \sigma_{N-1} \int_0^1 h^{N-1}(x_1)\, |\xi'(x_1)|^2 \, dx_1 = \mathrm{C}_0, \tag{A.22}$$

where $\xi(x_1)$ is the solution of the boundary value problem

$$\begin{cases} (h^{N-1}\, \xi')' = 0, & \text{in } (0,1), \\ \xi(0) = c_1^0, \quad \xi(1) = c_2^0. \end{cases} \tag{A.23}$$

where c_1^0 and c_2^0 are defined in Theorem 1.1.

Proof. From the variational characterization, we know that

$$\lambda_2^\epsilon = \inf \left\{ \frac{\displaystyle\int_{\Omega_\epsilon} |\nabla\varphi|^2\, dx}{\displaystyle\int_{\Omega_\epsilon} |\varphi|^2\, dx} \,:\, \varphi \in H^1(\Omega_\epsilon),\, \varphi \neq 0,\, \int_{\Omega_\epsilon} \varphi \quad dx = 0 \right\}.$$

Defining $\widetilde{\psi}(x_1, x')$ by

$$\widetilde{\psi}(x_1, x') = \begin{cases} \xi(x_1) & \text{in} \quad R_\epsilon \\ c_1^0 & \text{in} \quad \Omega^{\mathrm{L}} \\ c_2^0 & \text{in} \quad \Omega^{\mathrm{R}} \end{cases}$$

and $\psi(x_1, x') = \widetilde{\psi}(x_1, x') - \dfrac{1}{|\Omega_\epsilon|}\displaystyle\int_{\Omega_\epsilon} \widetilde{\psi}(x_1, x')\ dx_1\ dx'$, since $\widetilde{\psi} \in H^1(\Omega_\epsilon)$, then $\psi \in H^1(\Omega_\epsilon)$ and $\displaystyle\int_{\Omega_\epsilon} \psi\, dx_1\, dx' = 0$. So we have

$$\lambda_2^\epsilon \leqslant \frac{\displaystyle\int_{\Omega_\epsilon} |\nabla\psi|^2 dx}{\displaystyle\int_{\Omega_\epsilon} |\psi|^2 dx} = \frac{\sigma_{N-1}\,\epsilon^{N-1}\displaystyle\int_0^1 h^{N-1}(x_1)\,|\xi'(x_1)|^2\, dx_1\, dy'}{1 + \sigma_{N-1}\,\epsilon^{N-1}\displaystyle\int_0^1 h^{N-1}(x_1)|\xi(x_1)|^2 dx_1 + O(\epsilon^{2(N-1)})}$$

and consequently

$$\limsup_{\epsilon\to 0} \frac{\lambda_2^\epsilon}{\sigma_{N-1}\epsilon^{N-1}} \leqslant \int_0^1 h^{N-1}(x_1)\,|\xi'(x_1)|^2\, dx_1. \tag{A.24}$$

Now remains to show

$$\liminf_{\epsilon\to 0} \frac{\lambda_2^\epsilon}{\sigma_{N-1}\epsilon^{N-1}} \geqslant \int_0^1 h^{N-1}(x_1)\,|\xi'(x_1)|^2\, dx_1. \tag{A.25}$$

For this, we consider the following family of functions $\xi_\epsilon(x_1, y') = \varphi_2^\epsilon(x_1, \epsilon y')$, where $(x_1, y') \in R_1 = \{(x_1, y') : 0 \leqslant x_1 \leqslant 1,\, |y'| < h(x_1)\}$. If $x' = \epsilon y'$,

$$\epsilon^{N-1}\int_{R_1}\left[\left(\frac{\partial\xi_\epsilon}{\partial x_1}\right)^2 + \frac{1}{\epsilon^2}\left|\nabla_{y'}\xi_\epsilon\right|^2\right] dx_1\, dy' = \int_{R_\epsilon} |\nabla\varphi_2^\epsilon|^2\, dx_1\, dx' \leqslant \lambda_2^\epsilon. \tag{A.26}$$

Using (A.24) and (A.26) we have

$$\sup_{\epsilon>0}\int_{R_1}\left[\left(\frac{\partial\xi_\epsilon}{\partial x_1}\right)^2 + \frac{1}{\epsilon^2}\left|\nabla_{y'}\xi_\epsilon\right|^2\right] dx_1\, dy' < \infty. \tag{A.27}$$

It follows that the family of functions $\{\xi_\epsilon\}_{\epsilon>0}$ is uniformly bounded in $H^1(R_1)$. So we can find a subsequence $\{\epsilon_n\}$ with $\epsilon_n \overset{n\to\infty}{\longrightarrow} 0$, and a function $\xi_0 \in H^1(R_1)$ such that $\xi_{\epsilon_n} \overset{n\to\infty}{\longrightarrow} \xi_0$ weakly in $H^1(R_1)$ and strongly in $H^s(R_1)$ for all $s < 1$. Hence, from (A.27) we conclude that ξ_0 is independent of x'.

Let $\mathcal{K}$ be the convex and closed set of $H^1(0,1)$ defined by

$$\mathcal{K} = \left\{u \in H^1(0,1) : u(0) = c_1^0,\ u(1) = c_2^0\right\}.$$

Now we show that $\xi_0 \in \mathcal{K}$. In fact, since $\xi_0 \in H^1(R_1)$, we conclude immediately that $\xi_0 \in H^1(0,1)$. To find the boundary value of ξ_0 we use the fact that $\varphi_2^\epsilon \overset{\epsilon\to 0}{\longrightarrow} \phi_2$ strongly in $H^1(\Omega)$. Thus, from the continuity of the trace operator, it follows that

$$\xi_{\epsilon_n}|_{\partial R_1 \cap \Omega} \longrightarrow \xi_0|_{\partial R_1\cap\Omega} \quad \text{in} \quad H^{1/2}(\partial R_1).$$

Thus $\xi_0(0) = c_1^0$ and $\xi_0(1) = c_2^0$, which proves that $\xi_0 \in \mathcal{K}$.

Note that, from (A.26),

$$\lambda_2^\epsilon \geqslant \epsilon^{N-1} \int_{R_1} \left(\frac{\partial \xi_\epsilon}{\partial x_1}\right)^2 dx_1\, dy' \tag{A.28}$$

and that, if $J : \mathcal{K} \to \mathbb{R}$ is given by

$$J(u) = \int_0^1 h^{N-1}(x_1) u'(x_1)^2 dx_1,$$

then $J(\xi) = \min\limits_{u\in \mathrm{K}} J(u)$. Hence $J(\xi) \leqslant J(\xi_0)$. From (A.28), it follows that

$$\liminf_{n\to\infty} \frac{\lambda_2^{\epsilon_n}}{\epsilon^{N-1}} \quad\geqslant\quad \liminf_{\epsilon_n\to 0} \int_{R_1}\left(\frac{\partial \xi_{\epsilon_n}}{\partial x_1}\right)^2 dx_1\, dy' \geqslant \sigma_{N-1}\int_0^1 h^{N-1}(x_1)|\xi'(x_1)|^2 dx_1.$$

This gives (A.25), which along with (A.24) implies (A.22).

Replacing

$$\xi(x_1) = c_1^0 + (c_2^0 - c_1^0)\left\{\int_0^{x_1} \frac{dt}{h^{N-1}(t)}\right\}\left\{\int_0^1 \frac{dx_1}{h^{N-1}(x_1)}\right\}^{-1} \tag{A.29}$$

it on the righthand side of (A.24), C_0 appears. ∎

Theorem A.2 (Convergence of Eigenfunctions). Let $n \in \mathbb{N}$ and φ_n^ϵ eigenfunctions for problem (1.3), then

(i) $\varphi_1^\epsilon \to \phi_1$ in $H^k(\Omega)$ as $\epsilon \to 0$, $k \geqslant 1$,
(ii) $\sup\limits_\epsilon \|\varphi_2^\epsilon\|_{L^\infty(\Omega)} < \infty$ and $\varphi_2^\epsilon \to \phi_2$ in $H^1(\Omega)$ as $\epsilon \to 0$.

Proof. (i) Since that $\varphi_1^\epsilon = |\Omega_\epsilon|^{-1/2}$ and $\phi_1 = |\Omega|^{-1/2} = 1$, we have that $\varphi_1^\epsilon \to \phi_1$ in $H^k(\Omega)$ for all integer $k \geqslant 0$.

(ii) Let λ_2^ϵ be the second eigenvalue of (1.3) and φ_2^ϵ be a corresponding normalized eigenfunction. From Lemma A.1 and Lemma B.1 in Arrieta, Carvalho and Rodriguez-Bernal (2000), we have that $\varphi_2^\epsilon \in L^\infty(\Omega_\epsilon)$ and

$$\|\varphi_2^\epsilon\|_{L^\infty(\Omega_\epsilon)} \leqslant C\,\|\varphi_2^\epsilon\|_{L^2(\Omega_\epsilon)}$$

for some constant $C = C(|\Omega_\epsilon|, N)$.

This implies $\displaystyle\int_{R_\epsilon} |\varphi_2^\epsilon|^i \overset{\epsilon\to 0}{\longrightarrow} 0$, $i = 1, 2$. Hence, $\displaystyle\int_\Omega |\varphi_2^\epsilon|^2 \overset{\epsilon\to 0}{\longrightarrow} 1$ and $\displaystyle\int_\Omega \varphi_2^\epsilon \overset{\epsilon\to 0}{\longrightarrow} 0$.

Also, since

$$\int_\Omega |\nabla\varphi_2^\epsilon|^2 \leqslant \int_{\Omega_\epsilon} |\nabla\varphi_2^\epsilon|^2 = \lambda_2^\epsilon, \tag{A.30}$$

we have that

$$\int_\Omega |\nabla\varphi_2^\epsilon|^2 \overset{\epsilon\to 0}{\longrightarrow} 0. \tag{A.31}$$

Since $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, φ_2^ϵ has a convergent subsequence in $L^2(\Omega)$, which we denote again by φ_2^ϵ. Let φ be its limit. Thus we have

$$\begin{cases} \varphi_2^\epsilon \overset{\epsilon\to 0}{\longrightarrow} \varphi, & \text{strongly in} \quad L^2(\Omega), \\ \nabla\varphi_2^\epsilon \overset{\epsilon\to 0}{\longrightarrow} 0, & \text{strongly in} \quad L^2(\Omega). \end{cases}$$

Hence, $\displaystyle\int_\Omega \varphi = 0$, $\displaystyle\int_\Omega \varphi^2 = 1$, $\nabla\varphi = 0$ and $\varphi = c_1\chi_{\Omega^{\mathrm{L}}} + c_2\chi_{\Omega^{\mathrm{R}}}$ where c_1 and c_2 must satisfy

$$\begin{cases} c_1|\Omega^{\mathrm{L}}| + c_2|\Omega^{\mathrm{R}}| & = 0 \\ (c_1)^2|\Omega^{\mathrm{L}}| + (c_2)^2|\Omega^{\mathrm{R}}| & = 1. \end{cases} \tag{A.32}$$

Thus, $c_1 = \pm\sqrt{\frac{|\Omega^{\mathrm{R}}|}{|\Omega^{\mathrm{L}}|}}$ and $c_2 = \mp\sqrt{\frac{|\Omega^{\mathrm{L}}|}{|\Omega^{\mathrm{R}}|}}$. So, $\displaystyle\int_\Omega \varphi_2^\epsilon\phi_2 \overset{\epsilon\to 0}{\longrightarrow} \int_{\Omega^{\mathrm{L}}} c_1c_1^0 + \int_{\Omega^{\mathrm{R}}} c_2c_2^0$. From the first equation of (A.32) we see that c_1 and c_2 have opposite signs. We consider c_1 with negative sign and c_2 with positive sign. In this case $c_1 = c_1^0$ and $c_2 = c_2^0$. Thus we have

$$\int_\Omega \varphi_2^\epsilon\phi_2 \overset{\epsilon\to 0}{\longrightarrow} (c_1^0)^2|\Omega^{\mathrm{L}}| + (c_2^0)^2|\Omega^{\mathrm{R}}| = 1.$$

Now we show $\varphi_2^\epsilon \overset{\epsilon\to 0}{\longrightarrow} \phi_2$. In fact, $\|\varphi_2^\epsilon - \phi_2\|^2_{L^2(\Omega)} \leqslant 2 - 2\displaystyle\int_\Omega \varphi_2^\epsilon\phi_2 \overset{\epsilon\to 0}{\longrightarrow} 0$. From this and (A.31) we conclude that $\|\varphi_2^\epsilon - \phi_2\|_{H^1(\Omega)} \overset{\epsilon\to 0}{\longrightarrow} 0$. ∎

Finally, to conclude the proof of Theorem 1.1 we need to ensure that λ_3^ϵ is bounded away from zero.

Proposition A.3.

$$\liminf_{\epsilon\to 0} \lambda_3^\epsilon > 0.$$

Proof. Note that φ_3^ϵ satisfies

$$\int_{\Omega_\epsilon} \varphi_3^\epsilon = 0, \quad \int_{\Omega_\epsilon} \varphi_3^\epsilon \varphi_2^\epsilon = 0, \quad \|\varphi_3^\epsilon\|_{L^2(\Omega_\epsilon)} = 1. \tag{A.33}$$

Suppose that there is a sequence $\{\epsilon_j\}_{j=1}^\infty$ with $\epsilon_j \overset{j\to\infty}{\longrightarrow} 0$ such that $\lim_{j\to\infty} \lambda_3^{\epsilon_j} = 0$. Then

$$\int_\Omega |\nabla \varphi_3^{\epsilon_j}|^2 \leqslant \int_{\Omega_{\epsilon_j}} |\nabla \varphi_3^{\epsilon_j}|^2 = \lambda_3^{\epsilon_j} \|\varphi_3^{\epsilon_j}\|^2 = \lambda_3^{\epsilon_j} \overset{j\to\infty}{\longrightarrow} 0.$$

Hence there are $\varphi \in H^1(\Omega)$ and a subsequence of $\{\varphi_3^{\epsilon_j}\}$, with $\varphi_3^{\epsilon_j} \overset{j\to\infty}{\longrightarrow} \varphi$ weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$. Thus we have

$$\varphi_3^{\epsilon_j} \overset{s-L^2(\Omega)}{\longrightarrow} \varphi, \quad \nabla \varphi_3^{\epsilon_j} \overset{s-L^2(\Omega)}{\longrightarrow} 0.$$

It follows that $\varphi = c_1 \chi_{\Omega^{\mathrm{L}}} + c_2 \chi_{\Omega^{\mathrm{R}}}$. Now $\sup_{j\in\mathbb{N}} \lambda_3^{\epsilon_j} < \infty$ and Lemma B.1 in Arrieta, Carvalho and Rodriguez-Bernal (2000) imply that $\sup_{j\in\mathbb{N}} \|\varphi_3^{\epsilon_j}\|_{L^\infty(\Omega_{\epsilon_j})} < \infty$. This together with (A.33) imply that c_1 and c_2 satisfy

$$\begin{cases} c_1|\Omega^{\mathrm{L}}| + c_2|\Omega^{\mathrm{R}}| & = 0 \\ c_1 c_1^0 |\Omega^{\mathrm{L}}| + c_2 c_2^0 |\Omega^{\mathrm{R}}| & = 0 \\ (c_1)^2|\Omega^{\mathrm{L}}| + (c_2)^2|\Omega^{\mathrm{R}}| & = 1. \end{cases} \tag{A.34}$$

Since the above system does not have a solution, we have a contradiction and the result is proven. ∎

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- RESUMO: Neste artigo estudamos equações parabólicas em domínios do tipo Dumbbell. Isto é feito no caso da difusibilidade grande e condições de fronteiras não lineares.

- PALAVRAS-CHAVE: Domínio tipo Dumbbell; variedades invariantes; equilíbrios estáveis não constantes.

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