SUBCLASSES OF A TRAIN-ALGEBRA OF RANK 3

Roseli ARBACH[1]
Luis Antônio de Oliveira FERNANDES[1]

▪ *ABSTRACT: In this paper, we introduce, for t-algebras of rank 3, the notions of exceptional, normal and cancelative t-algebras of rank 3 (by analogy with Bernstein algebras). In sequence, some results are proved.*

▪ *KEYWORDS: t-algebras; Peirce decomposition; exceptional, normal and cancelative t-algebras of rank 3.*

1 Introduction

Let F be a field with $\text{char}(F) \neq 2$ and A be an F-algebra, not necessarily associative. If $\omega : A \to F$ is a nonzero homomorphism, then the ordered pair (A, ω) is called a *baric algebra* over F and ω its *weight function.* For each $x \in A$, $\omega(x)$ is called the *weight of x.* Let (A, ω) be a commutative baric algebra. If there exist $\gamma_1, \gamma_2, \ldots, \gamma_{n-1} \in F$ such that, for all x in A,

$$x^n + \gamma_1 \omega(x) x^{n-1} + \ldots + \gamma_{n-1} \omega(x)^{n-1} x = 0 \qquad (1)$$

we say that (A, ω) is a *(commutative) train algebra* (in short, *t-algebra*). Moreover, if there is no similar relation involving $x^{n-1}, \ldots, x$, we say that n is the rank of (A, ω) and (1) is its *train equation* (in short, *t-equation*). In this case, it is easy to see that $1 + \gamma_1 + \gamma_2 + \ldots + \gamma_{n-1} = 0$. Moreover, $x^n = 0$ for all $x \in Ker(\omega)$. If (A, ω) is a commutative t-algebra of rank n, then its t-equation is unique. In this paper, we will consider only t-algebras of rank 3, that is, those satisfying an equation

$$x^3 - (1 + \gamma) \omega(x) x^2 + \gamma \omega(x)^2 x = 0 \qquad (2)$$

[1]Departamento de Matemática, Faculdade de Engenharia de Ilha Solteira, Universidade Estadual Paulista – UNESP, Caixa Postal 31, CEP: 15 385-000, Ilha Solteira, São Paulo, Brasil. E-mail: *roseli@mat.feis.unesp.br* / *lafo@mat.feis.unesp.br*

where $\gamma \in F$ is a fixed element. For details about t-algebras of rank 3, the reader is referred to Costa (1990), Arbach (1997) and Wőrz (1980). In this paper we assume that $2\gamma \neq 1$ and, as a consequence, there is (at least) an idempotent $e \in A$ and relative to this element, A has a Peirce decomposition $A = Fe \oplus U_e \oplus V_e$ in which, denoting $Ker(\omega)$ by N,

$$U_e = \{u \in N : 2eu = u\} \tag{3}$$

$$V_e = \{v \in N : ev = \gamma v\} \tag{4}$$

$$N = U_e \oplus V_e \tag{5}$$

Decomposition $A = Fe \oplus U_e \oplus V_e$ depends on the choice of the idempotent in A, but it can be proved that the dimensions of U_e and V_e are invariants, see Costa (1990). Then we can define the invariant *type of* A as the ordered pair of integers $(1 + r, s)$, where $r = dim(U_e)$ and $s = dim(V_e)$. The subspaces U_e and V_e satisfy the following relations: $U_e^2 \subseteq V_e$; $U_eV_e \subseteq U_e$; $V_e^2 = 0$; $U_e^{2n+1} \subseteq U_e$ $(n \geq 0)$ and $U_e^{2n} \subseteq V_e$ $(n \geq 1)$. Moreover, by the second linearization of (2), for all elements $x, y, z \in N$, we obtain the Jacobi's identity

$$J(x, y, z) = x(yz) + y(xz) + z(xy) = 0 \tag{6}$$

Let $I(A)$ be the set of the idempotents elements of A. It follows of the above relations that $I(A) = \{e_0 = e + u_0 + \lambda u_0^2 : u_0 \in U_e\}$, where $\lambda = (1 + 2\gamma)^{-1}$. We consider the Peirce decomposition of A in relation to the idempotent e_0, that is, $A = Fe_0 \oplus U_0 \oplus V_0$, where $U_0 = \{u \in N : 2e_0u = u\}$ and $V_0 = \{v \in N : e_0v = \gamma v\}$. The subspaces U_e an U_0, V_e and V_0 are related by $U_0 = \{u + 2\lambda\, u_0u : u \in U_e\}$ and $V_0 = \{v - 2\lambda\, u_0v : v \in V_e\}$.

Now we will explore a result that appears shortly mentioned in Guzzo Júnior and Vicente (1998). For that, consider $A = Fe \oplus U_e \oplus V_e$ a t-algebra of rank 3. In order to simplify notation, we use U and V in place of U_e and V_e, respectively. To each fixed $\gamma \in F$, $2\gamma \neq 1$, there is a class of t-algebras of rank 3 that satisfy the equation $x^3 - (1 + \gamma)\omega(x)x^2 + \gamma\omega(x)x = 0$. If A satisfies the equation $x^3 - \omega(x)x^2 = 0$ (that is, if $\gamma = 0$), it is possible to pass to another class in which $\gamma \neq 0$ if we define over the F-vectorial space A a new multiplication, as follows:

$$x \circ y = (1 - 2\gamma)xy + \gamma[x\omega(y) + y\omega(x)] \tag{7}$$

with $\gamma \in F$ different from $\frac{1}{2}$. We have the following results about this new algebra (which will be denoted by A_1):

Lemma 1.1. *Let $A = Fe \oplus U \oplus V$ be a t-algebra of rank 3 that satisfies the t-equation $x^3 - \omega(x)x^2 = 0$. Consider $\gamma \in F$, with $2\gamma \neq 1$ and A_1 the algebra defined as above. Then* *(a) A_1 is baric, commutative, with weight function ω;*

(b) A and A_1 have the same idempotents;

(c) A_1 satisfies the t-equation $x^3 - (1+\gamma)\omega(x)x^2 + \gamma\omega(x)^2 x = 0$;

(d) The Peirce decomposition of A_1, relative to the idempotent e, is $A_1 = Fe \oplus U \oplus V$.

Proof: To prove (a) and (c), see Guzzo Júnior and Vicente (1998).

(b) If $e \in A$ is an idempotent element, then $e \circ e = (1-2\gamma)e^2 + 2\gamma\omega(e)e = (1-2\gamma)e + 2\gamma e = e$. Conversely, $e \circ e = e$ implies $(1-2\gamma)e^2 = e \circ e - 2\gamma e = (1-2\gamma)e$, that is, $e^2 = e$.

(d) Consider $U^* = \{u \in N : 2\ e \circ u = u\}$ and $V^* = \{v \in N : e \circ v = \gamma v\}$. If $u \in U^*$, then $u = u_1 + v_1$, with $u_1 \in U$, $v_1 \in V$ and so $u = 2\ e \circ u = 2\ e \circ (u_1 + v_1) = 2\ e \circ u_1 + 2\ e \circ v_1 = 2[(1-2\gamma)eu_1 + \gamma u_1] + 2[(1-2\gamma)ev_1 + \gamma v_1] = (1-2\gamma)u_1 + 2\gamma u_1 + 2\gamma v_1 = (1-2\gamma)u_1 + 2\gamma u$. Therefore, $(1-2\gamma)u = (1-2\gamma)u_1$ and this means that $u = u_1$ and so $U^* \subseteq U$. Conversely, if $u \in U$, then $2e \circ u = 2[(1-2\gamma)eu + \gamma u] = 2eu = u$ so $u \in U^*$ and this implies $U^* = U$. In the same way it can be proved that $V^* = V$. ∎

2 Definitions and examples

In the Bernstein algebras theory, the subclasses of the *nuclear*, *exceptional*, *normal* and *cancelative algebras* are known. By analogy, we extend those ideas to t-algebras of rank 3. For that, consider $A = Fe \oplus U \oplus V$ a t-algebra of rank 3, with $\gamma \neq 0$. Observe that $A^2 = A$, since if $x = ke + u + v \in A$, and so $x = ke^2 + 2eu + \gamma^{-1}ev \in A^2$ and it is not necessary to define nuclear t-algebras of rank 3. In the following, we use $< w_1, w_2, \ldots, w_n >$ to denote the vectorial subspace generated by vectors $w_1, w_2, \ldots, w_n$ in some vectorial space.

Definition 2.1. A t-algebra of rank 3 (A, ω) satisfying (2) is called a
(i) *exceptional t-algebra of rank 3* if $U^2 = 0$, for some Peirce decomposition of A;
(ii) *normal t-algebra of rank 3* if $UV = 0$, for some Peirce decomposition of A.

A simple computation shows that algebra $A_1 = < e, u_1, u_2, v_1, v_2 >$, whose multiplication table is $e^2 = e$, $2eu_i = u_i$ $(i = 1,2)$, $ev_i = v_i$ $(i = 1,2)$, $u_1v_2 = u_2$ and the other products are zero is a exceptional t-algebra of rank 3. Moreover, the algebra $A_2 = < e, u, u^2, v >$, which multiplication table is $e^2 = e, 2eu = u, u^2 \neq 0$ and the other products are zero is a t-algebra of rank 3, but A_2 is not an exceptional one. In the same way, A_2 is a normal t-algebra of rank 3 and A_1 is a t-algebra of rank 3, but it is not a normal one.

Lemma 2.2. *Let $A = Fe \oplus U \oplus V$ be a t-algebra of rank 3 satisfying (2). Then* *(i) If A is an exceptional t-algebra, then A is an exceptional t-algebra, related to any idempotent element $e_0 \in I(A)$;*

(ii) If A is a normal t-algebra, then A is a normal t-algebra, related to any idempotent element $e_0 \in I(A)$.

Proof: (i) Since $e_0 \ \in I(A)$, there is $u_0 \ \in U$ such that $e_0 = e + u_0 + \lambda u_0^2$ and so $U_0 = \{u + 2\lambda u_0 u : u \in U\} = U$, as $u_0 u \in U^2 = 0$.

(ii) It is enough to see that, if $e_0 = e + u_0 + \lambda u_0^2 \ \in I(A)$, then a generic generator of $U_0 V_0$ is $(u + 2\lambda u_0 u)v = uv + 2\lambda(u_0 u)v = uv = 0$, since $(u_0 u)v \in$ $\in V^2 = 0$. ∎

The next Proposition gives a characterization for normal t-algebras of rank 3.

Proposition 2.3. *$A = Fe \oplus U \oplus V$ is a normal t-algebra of rank 3 satisfying (2), then for all $x, y \in A$*

$$x^2 y - \omega(x)xy - \gamma\, \omega(y)[x^2 - \omega(x)x] = 0 \tag{8}$$

Proof: Let $x = ke + u + v$ and $y = k'e + u' + v'$ be two arbitrary elements in a normal t-algebra of rank 3 A. It is easy to see that

$$2x^2 = 2k^2 e + 2ku + 2u^2 + 4\gamma kv$$
$$2xy = 2kk'e + k'u + ku' + 2\gamma k'v + 2\gamma kv' + 2uu'$$
$$2x^2 y = 2k^2 k'e + kk'u + k^2 u' + 2\gamma k'u^2 + 4\gamma^2 kk'v + 2kuu' + 2\gamma k^2 v'$$

and so, by a simple computation, we prove (8).
Conversely, consider $uv \in UV$ and let, in (8), $x = u + v$ and $y = e$. Then

$$(u+v)^2 e - \omega(u+v)[(u+v)e] - \gamma\omega(e)[(u+v)^2 - w(u+v)(u+v)] = 0$$

By $\omega(u + v) = 0$, we have

$$(u^2 + 2uv + v^2)e - \gamma(u^2 + 2uv + v^2) = 0$$

and, by $eu^2 = \gamma u^2$, $2e(uv) = uv$ and $v^2 = 0$, it follows that

$$\gamma u^2 + uv - \gamma u^2 + 2\gamma uv = 0$$

That is, $(1 - 2\gamma)uv = 0$ and by $1 - 2\gamma \neq 0$, we have $uv = 0$. ∎

Definition 2.4. A t-algebra of rank 3 $A = Fe \oplus U \oplus V$ is called a *cancelative* t-algebra if, for all $0 \neq u \in U$, the linear application $L_{\mathrm{u}} : U \longrightarrow V$ defined by $L_{\mathrm{u}}(u_1) = uu_1$, is an injective application.

The algebra of Example A_2 is a cancelative t-algebra of rank 3. Consider now algebra $A_3 =< e, u_1, u_2, u_1u_2, v >$, whose multiplication table is $e^2 = e$, $2eu_i = u_i$ $(i = 1, 2)$, $u_1u_2 \neq 0$ and the other products are zero. Then A_3 is a t-algebra of rank 3 (satisfying the t-equation $x^3 - \omega(x)^2x = 0$). Moreover, L_{u_1} is not injective, since $u_1 \neq 0$ and, for example, $L_{u_1}(u_1) = 0$ and so A_3 is not an exceptional t-algebra of rank 3.

Proposition 2.5. *If $A = Fe \oplus U \oplus V$ is a cancelative t-algebra of rank 3, then A is a normal t-algebra of rank 3.*

Proof: Let $0 \neq u \in U$ and $v \in V$ be arbitrary elements. In the Jacobi's identity, by considering $x = y = u$ and $z = v$, we have $2u(uv) + u^2v = 0$ and since $u^2v \in V^2 = \{0\}$, it follows that $L_u(uv) = u(uv) = 0$. But A is cancelative t-algebra and $u \neq 0$, and so $uv = 0$. By linearity, we get $UV = 0$ and so A is normal. ■

Consider now algebra $A_4 = < e, u_1, u_2, u_1^2, u_2^2 >$, whose multiplication table is $e^2 = e, 2eu_i = u_i$ $(i = 1, 2), u_i^2 \neq 0$ $(i = 1, 2)$, and the other products are zero. It is easy to prove that A_4 is t-algebra of rank 3, satisfying the t-equation $x^3 - \omega(x)^2x = 0$. By the multiplication table, we see that $UV = 0$ and so A_4 is a normal t-algebra. Moreover, $u_i \neq 0$, for $i = 1, 2$, and $L_{u_1}(u_2) = u_1u_2 = 0$, that is, A_4 is not a cancelative t-algebra.

Lemma 2.6. *If $A = Fe \oplus U \oplus V$ is a cancelative t-algebra of rank 3, then A is a cancelative t-algebra, related to any idempotent element $e_0 \in I(A)$.*

Proof: Let $e_0 = e + u_0 + \lambda u_0^2 \in I(A)$ be an idempotent element of A and U_0 the corresponding Peirce's subspace. Consider now $0 \neq u' = u + \lambda u_0u$ and $u_1' = u_1 + \lambda u_0u_1 \in U_0$ such that $L_{u'}(u_1') = u'u_1' = 0$. Since A is a cancelative algebra, by Proposition 2.5, A is a normal one and so $u_0(uu_1) \in UV = 0$. Then $0 = (u + \lambda u_0u)(u_1 + \lambda u_0u_1) = uu_1 = L_u(u_1)$ and so $u_1 = 0$, which implies $u_1' = 0$. ■

3 Some results about cancelatives t-algebras of rank 3

A Jordan algebra is a commutative non-associative algebra A that satisfies the identity $(x^2y)x - x^2(yx) = 0$, for all $x, y \in A$.

Theorem 3.1. *If A is a cancelative t-algebra of rank 3 satisfying the train equation (2), then the following conditions are equivalent:*

(i) *A is a Jordan algebra;*

(ii) $\gamma = 0$ *or* $\gamma = 1$.

Proof: By Proposition 2.5, A is a normal algebra and, by Proposition 2.3, satisfies (8) and so, $(x^2y)x = \omega(x)x(xy) + \gamma\ \omega(y)[x^3 - \omega(x)x^2]$. By linearizing this last identity, we obtain

$$2(xz)y - \omega(z)xy - \omega(x)yz - \gamma\omega(y)[2xz - \omega(z)x - \omega(x)z] = 0 \tag{9}$$

and for $x, z = x$ and $y = yx$, we have $x^2(yx) = \omega(x)x(xy) + \gamma\omega(xy)[x^2 - \omega(x)x]$ Now, we can prove the equivalences. For that, consider $x, y \in A$. Then

$$(x^2y)x = x^2(yx) \iff \gamma\ \omega(y)[x^3\ -\ \omega(x)x^2] = \gamma\omega(xy)[x^2 - \omega(x)x] \iff$$

$$\iff \gamma\ \omega(y)[x^3\ -\ 2\omega(x)x^2 + \omega(x)^2x] = 0$$

Consider, now, $y \in A$ such that $\omega(y) = 1$. Then
$(x^2y)x = x^2(yx) \iff \gamma[x^3 - 2\omega(x)x^2 + \omega(x)^2x] = 0 \iff \gamma = 0$ or $x^3 - 2\omega(x)x^2 + \omega(x)^2x = 0 \iff \gamma = 0$ or $\gamma = 1$. ∎

Proposition 3.2. *If A is a cancelative t-algebra of rank 3 of type $(1 + r, s)$, then $0 \leq r \leq s$.*

Proof: If $r = 0$, then the result is valid. Suppose, now, $r > 0$. Then $U \neq 0$ and we can consider $0 \neq u \in U$. Since A is a cancelative algebra, the linear application $L_u : U \longrightarrow V$ is injective and so $r = dim(U) \leq dim(V) = s$. ∎

Theorem 3.3. *Let A be a t-algebra of rank 3 of type $(1 + r, s)$, $r \geq 1$, satisfying $2dim(U^2) = r(r + 1)$. Then A is a cancelative t-algebra.*

Proof: Consider $0 \neq u \in U$ and $u_0 \in U$ such that $uu_0 = 0$. Let $B_U = \{u = = u_1, u_2, \ldots, u_r\}$ be a basis of U. Then $U^2 = [u_iu_j]$, for $1 \leq i, j \leq r$. Since A is a commutative algebra, there are $\frac{1}{2}r(r + 1) = dim(U^2)$ distinct elements of type u_iu_j, for $1, \leq i, j \leq r$ and so these elements form a basis for U^2. Otherwise, $u_0 \in U$ and so there are $\alpha_1, \alpha_2, \ldots, \alpha_r$ such that

$$u_0 = \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_r u_r = \sum_{i=1}^{r} \alpha_i u_i$$

and so

$$0 = uu_0 = u_1u_0 = u_1 \sum_{i=1}^{r} \alpha_i u_i = \sum_{i=1}^{r} \alpha_i u_1 u_i$$

Since the elements $u_1u_i(1 \leq i \leq r)$ are free, it follows that $\alpha_i = 0$ (for $i = 1, 2, \ldots, r$) and so $u_0 = 0$. Then L_u is a injective application, for all $0 \neq u \in U$; that is, A is a cancelative algebra. ∎

ARBACH, R.; FERNANDES, L. A. O. Subclasses de uma t-algebra de posto 3. *Rev. Mat. Estat.,* São Paulo, v.25, n.1, p.23-29, 2007.

- RESUMO: Neste artigo, por analogia com álgebras de Bernstein, introduzimos para as t-álgebras de posto 3, as noções de t-álgebras de posto 3 *excepcionais*, *normais* e *cancelativas*. Em seguida, provamos alguns resultados sobre estas álgebras.

- PALAVRAS-CHAVE: t-álgebras; decomposição de Peirce; t-álgebras de posto 3 excepcionais, normais e cancelativas.

References

ARBACH, R. *Sobre os P-subespaços em uma train álgebra de posto 3*. 1997. 136f. Thesis (Ph.D.) - Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 1997.

COSTA, R. Principal t-algebras of rank 3 and dimension ≤ 5. *Proc. Edinb. Math. Soc.*, Edinburgh, v.33, p.61-70, 1990.

GUZZO JÚNIOR, H.; VICENTE P. On Bernstein and train algebras of rank 3. *Comm. Algebra*, New York, v.26, n.7, p.2021-2032, 1998.

WŐRZ, A. *Algebras in genetics*. Berlin, 1980. 237p. (Lecture Notes in Biomathematics, v.36).

Received in 09.05.2006.

Approved after revised in 08.01.2007.