WAITING TIME FOR A RUN OF *N* SUCCESSES IN BERNOULLI SEQUENCES

Lucas Monteiro CHAVES[1]
Devanil Jaques de SOUZA[1]

▪ *ABSTRACT: We use* j*-step Fibonacci numbers to derive the expected value and the variance for the distribution of the waiting time for* n *consecutive successes in a Bernoulli sequences. For the Bernoulli parameter* $p = 1/2$ *the exact distribution is obtained.*

▪ *KEYWORDS: Fibonacci numbers; waiting time distribution; Bernoulli sequences.*

1 Introduction

The study of runs, that is, consecutive occurrences of the same event in a random sequence, is a very important tool in non-parametric statistics. For example, run tests and rank sum tests, both based on counting runs of successes, are used to decide if two populations are equal (Mood, 1940). This is also related to random walk theory (Feller, 1968). In the case of Bernoulli trials there is a series of results (Greenberg, 1970; Saperstein, 1973; Koutras, 1996; Aki and Hirano, 2007). Another problem is the study of waiting time for the occurrence of runs with given width, related to the negative binomial distribution. In the same line, there are results about waiting time for patterns in the context of Markov Chains (Balasubramanian, Viveros and Balakrishnan, 1993; Koutras and Alexandrou, 1995).

In this article we use only elementary facts about *j*-step Fibonacci numbers to derive the expected value and the variance of the waiting time for *n* consecutive successes in Bernoulli sequences. In the case of the Bernoulli parameter $p = 1/2$ the exact distribution is obtained. It is also obtained the generating function for the *j*-step Fibonacci numbers. Authors couldn't find any reference for this last result.

2 Development

2.1 Expected value is easy for any *p*

Let $T^{(j)}$ be the waiting time for a run of j successes in a *p* parameter Bernoulli sequence. Let X be a *p* parameter Bernoulli random variable, independent of $T^{(j)}$.

[1]Departamento de Ciências Exatas, Universidade Federal de Lavras – UFLA, Postal Box: 37, CEP: 37200-000, Lavras, MG, Brazil. E-mail:*lucas@ufla.br* / *djs@oi.com.br*

Then,

$$T^{(j)} \overset{d}{=} X\left(T^{(j-1)}+1\right)+\left(1-X\right)\left(T^{(j-1)}+1+T^{(j)}\right)$$

$$\mathrm{E}\left(T^{(j)}\right) = p\mathrm{E}\left(T^{(j-1)}+1\right)+\left(1-p\right)\mathrm{E}\left(T^{(j-1)}+1+T^{(j)}\right)$$

$$= p\mathrm{E}\left(T^{(j-1)}\right)+p+\mathrm{E}\left(T^{(j-1)}\right)+1+\mathrm{E}\left(T^{(j)}\right)-p\mathrm{E}\left(T^{(j-1)}\right)-p-p\mathrm{E}\left(T^{(j)}\right)$$

$$= \mathrm{E}\left(T^{(j-1)}\right)+1+\mathrm{E}\left(T^{(j)}\right)-p\mathrm{E}\left(T^{(j)}\right)$$

$$= \frac{1}{p}\mathrm{E}\left(T^{(j-1)}\right)+\frac{1}{p}$$

We solve this recursive formula:

$$\mathrm{E}\left(T^{(j)}\right) = \frac{1}{p}\left[\frac{1}{p}\mathrm{E}\left(T^{(j-2)}\right)+\frac{1}{p}\right]+\frac{1}{p}$$

$$= \frac{1}{p^{2}}\mathrm{E}\left(T^{(j-2)}\right)+\frac{1}{p^{2}}+\frac{1}{p}$$

$$= \frac{1}{p^{3}}\mathrm{E}\left(T^{(j-3)}\right)+\frac{1}{p^{3}}+\frac{1}{p^{2}}+\frac{1}{p}$$

$$\ldots\ldots \qquad \ldots\ldots \qquad \ldots\ldots$$

$$= \frac{1}{p^{j-1}}\mathrm{E}\left(T^{(1)}\right)+\frac{1}{p^{j-1}}+\frac{1}{p^{j-2}}+\ldots+\frac{1}{p}$$

$$= \frac{\frac{1}{p}\left[\left(\frac{1}{p}\right)^{j}-1\right]}{\frac{1}{p}-1}$$

$$= \frac{1-p^{j}}{1-p}\frac{1}{p^{j}}$$

2.2 The variance is not so easy

$$\mathrm{Var}\left(T^{(j)}\right) = \left[\mathrm{E}\left(T^{(j)}\right)^{2}\right]-\left[\mathrm{E}\left(T^{(j)}\right)\right]^{2} \qquad (1)$$

$$\left[\mathrm{E}\left(T^{(j)}\right)^2\right] \quad = \mathrm{E}\left[X\left(T^{(j-1)}+1\right)+(1-X)\left(T^{(j-1)}+1+T^{(j)}\right)\right]^2$$

As either $X=0$ or $(1-X)=0$, it follows that

$$\begin{aligned}
\left[\mathrm{E}\left(T^{(j)}\right)^2\right] &= \mathrm{E}\left\{\left[X\left(T^{(j-1)}+1\right)\right]^2+\left[(1-X)\left(T^{(j-1)}+1+T^{(j)}\right)\right]^2\right\} \\
&= p\mathrm{E}\left[\left(T^{(j-1)}\right)^2+2T^{(j-1)}+1\right]+(1-p)\mathrm{E}\left[\left(T^{(j)}\right)^2+\left(T^{(j-1)}\right)^2\right. \\
&\qquad \left.+1+2T^{(j)}+2T^{(j-1)}+2T^{(j)}T^{(j-1)}\right] \\
&= p\mathrm{E}\left[\left(T^{(j-1)}\right)^2+2T^{(j-1)}+1\right]+(1-p)\mathrm{E}\left[\left(T^{(j-1)}\right)^2+2T^{(j-1)}+1\right]+ \\
&\qquad +(1-p)\mathrm{E}\left[\left(T^{(j)}\right)^2+2T^{(j)}+2T^{(j)}T^{(j-1)}\right] \\
&= \mathrm{E}\left[\left(T^{(j-1)}\right)^2+2T^{(j-1)}+1\right] \\
&\qquad +(1-p)\mathrm{E}\left[\left(T^{(j)}\right)^2+2T^{(j)}+2T^{(j)}T^{(j-1)}\right] \qquad (2)
\end{aligned}$$

$$\begin{aligned}
\left[\mathrm{E}\left(T^{(j)}\right)\right]^2 &= \left[p\mathrm{E}\left(T^{(j-1)}+1\right)+(1-p)\mathrm{E}\left(T^{(j-1)}+1+T^{(j)}\right)\right]^2 \\
&= \left[\mathrm{E}\left(T^{(j-1)}\right)+1+(1-p)\mathrm{E}\left(T^{(j)}\right)\right]^2 \\
&= \mathrm{E}^2\left(T^{(j-1)}\right)+1+(1-p)^2\mathrm{E}^2\left(T^{(j)}\right)+2\mathrm{E}\left(T^{(j-1)}\right)+ \\
&\qquad +2(1-p)\mathrm{E}\left(T^{(j)}\right)+2(1-p)\mathrm{E}\left(T^{(j)}\right)\left(T^{(j-1)}\right) \qquad (3)
\end{aligned}$$

Plugging (2) and (3) in (1):

$$\begin{aligned}
\mathrm{Var}\left(T^{(j)}\right) &= \mathrm{E}\left(T^{(j-1)}\right)^2-\mathrm{E}^2\left(T^{(j-1)}\right)+(1-p)\mathrm{E}\left(T^{(j)}\right)^2-(1-p)^2\mathrm{E}^2\left(T^{(j)}\right) \\
&= \mathrm{Var}\left(T^{(j-1)}\right)+(1-p)\left[\mathrm{E}\left(T^{(j)}\right)^2-\mathrm{E}^2\left(T^{(j)}\right)+p\mathrm{E}^2\left(T^{(j)}\right)\right] \\
&= \mathrm{Var}\left(T^{(j-1)}\right)+(1-p)\mathrm{Var}\left(T^{(j)}\right)+p(1-p)\mathrm{E}^2\left(T^{(j)}\right)
\end{aligned}$$

So, we get the recursive formula

$$\boxed{\mathrm{Var}\left(T^{(j)}\right)=\frac{1}{p}\mathrm{Var}\left(T^{(j-1)}\right)+(1-p)\mathrm{E}^2\left(T^{(j)}\right)}$$

and we solve it [2]

$$
\begin{aligned}
\mathrm{Var}\left(T^{(j)}\right) &= \frac{1}{p}\left\{\frac{1}{p}\mathrm{Var}\left(T^{(j-2)}\right)+(1-p)\left[\mathrm{E}\left(T^{(j-1)}\right)\right]^2\right\}+(1-p)\left[\mathrm{E}\left(T^{(j)}\right)\right]^2\\
&= \frac{1}{p^2}\mathrm{Var}\left(T^{(j-2)}\right)+\frac{(1-p)}{p}\left[\mathrm{E}\left(T^{(j-1)}\right)\right]^2+(1-p)\left[\mathrm{E}\left(T^{(j)}\right)\right]^2\\
&= \frac{1}{p^3}\mathrm{Var}\left(T^{(j-3)}\right)+\frac{(1-p)}{p^2}\left[\mathrm{E}\left(T^{(j-2)}\right)\right]^2+\\
&\qquad +\frac{(1-p)}{p}\left[\mathrm{E}\left(T^{(j-1)}\right)\right]^2+(1-p)\left[\mathrm{E}\left(T^{(j)}\right)\right]^2\\
&\qquad \ldots\ldots \qquad \ldots\ldots \qquad \ldots\ldots\\
&= \frac{1}{p^{j-1}}\mathrm{Var}\left(T^{(1)}\right)+(1-p)\sum_{i=0}^{j-2}\frac{1}{p^i}\left[\mathrm{E}\left(T^{(j-i)}\right)\right]^2\\
&= \frac{1}{p^{j-1}}\frac{1-p}{p^2}+(1-p)\sum_{i=0}^{j-2}\frac{1}{p^i}\left[\frac{1-p^{j-i}}{1-p}\frac{1}{p^{j-i}}\right]^2\\
&= \frac{1-p}{p^{j+1}}+\frac{1}{(1-p)p^{2j}}\sum_{i=0}^{j-2}p^i\left(1-p^{j-i}\right)^2\\
&= \frac{1-p}{p^{j+1}}-\frac{2(j-1)}{(1-p)p^{2j}}p^j+\frac{1}{(1-p)p^{2j}}\left[\frac{1-p^{j-1}}{1-p}+\frac{p^{2j}\left[1-\left(\frac{1}{p}\right)^{j-1}\right]}{1-\left(\frac{1}{p}\right)}\right]\\
&= \frac{1-p}{p^{j+1}}-\frac{2(j-1)}{(1-p)p^j}+\frac{1}{(1-p)p^{2j}}\left[\frac{1-p^{j-1}}{1-p}+\frac{p^{2j}\left(\frac{p^{j-1}-1}{p^{j-1}}\right)}{\frac{p-1}{p}}\right]\\
&= \frac{1-p}{p^{j+1}}-\frac{2(j-1)}{(1-p)p^j}+\frac{1-p^{j-1}}{(1-p)^2p^{2j}}\left(1+p^{j+2}\right)\\
&= -\frac{p-(1+2j)(p^{-j}-p^{1-j})-p^{-2j}}{\left(1-p\right)^2}
\end{aligned}
$$

[2] This problem was pointed out to us by Prof. Thomas Ferguson in a particular communication.

2.3 *j*-step Fibonacci numbers

The *j*-step Fibonacci numbers, denoted by $F_n^{(j)}$, are defined as

$$
\begin{aligned}
F_n^{(j)} &= 0 && \text{for } n \le 0 \\
F_1^{(j)} &= 1 \\
F_n^{(j)} &= F_{n-1}^{(j)} + F_{n-2}^{(j)} + \ldots + F_{n-j}^{(j)} && \text{for } n \ge 2
\end{aligned}
$$

For $j = 2$ those are just the traditional Fibonacci numbers

$$F_n^{(2)} = F_n = \{0, \quad 1, \quad 1, \quad 2, \quad 3, \quad 5, \quad 8, \quad 13, \quad 21, \ldots\}$$

For $j > 2$ we list some values:

$n =$	0	1	2	3	4	5	6	7	8
$F_n^{(3)} =$	0	1	1	2	4	7	13	24	44
$F_n^{(4)} =$	0	1	1	2	4	8	15	29	56
$F_n^{(5)} =$	0	1	1	2	4	8	16	31	61
$F_n^{(6)} =$	0	1	1	2	4	8	16	32	63

(4)

To obtain the generator function we write the power series:

$$
\begin{aligned}
P(x) &= F_0^{(j)} + x\,F_1^{(j)} + x^2\,F_2^{(j)} + x^3\,F_3^{(j)} + \ldots = \sum_{n=0}^{\infty} x^n\,F_n^{(j)} \\
x\,P(x) &= x\,F_0^{(j)} + x^2\,F_1^{(j)} + x^3\,F_2^{(j)} + x^4\,F_3^{(j)} + \ldots = \sum_{n=0}^{\infty} x^{n+1}\,F_n^{(j)} \\
x^2\,P(x) &= x^2\,F_0^{(j)} + x^3\,F_1^{(j)} + x^4\,F_2^{(j)} + x^5\,F_3^{(j)} + \ldots = \sum_{n=0}^{\infty} x^{n+2}\,F_n^{(j)} \\
&\ldots\ldots\ldots\ldots\ldots \\
x^j\,P(x) &= x^j\,F_0^{(j)} + x^{j+1}\,F_1^{(j)} + x^{j+2}\,F_2^{(j)} + x^{j+3}\,F_3^{(j)} + \ldots = \sum_{n=0}^{\infty} x^{n+j}\,F_n^{(j)}
\end{aligned}
$$

and take the differences

$$\left(1 - x - x^2 - ... - x^j\right) P(x) = F_0^{(j)} + x\,(F_1^{(j)} - F_0^{(j)}) + x^2\,(F_2^{(j)} - F_1^{(j)} - F_0^{(j)}) + $$
$$+ x^3\,(F_3^{(j)} - F_2^{(j)} - F_1^{(j)} - F_0^{(j)}) + ... + $$
$$+ \sum_{n=j}^{\infty} x^j \left(F_n^{(j)} - F_{n-1}^{(j)} - F_{n-2}^{(j)} - ... - F_{n-j}^{(j)}\right)$$
$$= 0 + x + x^2\,(0) + ... + 0 + \sum_{n=j}^{\infty} x^j\,(0)$$

$$\sum_{n=1}^{\infty} x^n\, F_n^{(j)} = \frac{x}{1 - x - x^2 - ... - x^j}$$

2.4 Bernoulli sequences with no runs of j consecutive successes

Let $S_k^{(j)}$ be the number of size *k* Bernoulli sequences (sequences of 0's and 1's) with no runs of *j* consecutive 1's (good sequences, from now on). To derive the value of $S_k^{(j)}$ we set, for convenience, $S_0^{(j)} = 1$ and we note that:

(a) For $k \in \{1, 2, ..., j-1\}$ $\quad S_k^{(j)} = 2^k$

(b) For $k = j$ $\quad S_j^{(j)} = 2^j - 1$ (all sequences except $\underbrace{1 \;\; 1 \;\; ... \;\; 1}_{j}$)

$$= S_{j-1}^{(j)} + S_{j-2}^{(j)} + ... + S_0^{(j)}$$

(c) For $k > j$, the number of good sequences that finish with a 0 is $S_{k-1}^{(j)}$ because any size (*k*-1) sequence without j consecutive 1's is allowed before the ending 0. Then we can partition the set of "success" sequences as follows

$$\overbrace{\_ \;\; \_ \;\; \_ \;\; \_ \;\; \_ \;\; \_ \;\; 0}^{k} \;\; \Rightarrow \;\; \text{number of sequences}=S_{k-1}^{(j)}$$

$$\overbrace{\_ \;\; \_ \;\; \_ \;\; \_ \;\; \_ \;\; 0 \;\; 1}^{k} \;\; \Rightarrow \;\; S_{k-2}^{(j)}$$

$$\overbrace{\_\ \ \_\ \ \_\ \ \_\ \ 0\ \ 1\ \ 1}^{k} \quad \Rightarrow \quad S_{k-3}^{(j)}$$

. . . .

$$\overbrace{\_\ \ \_\ \ \_\ \ \underbrace{0\ \ 1\ \ ....\ \ 1}_{j}}^{k} \quad \Rightarrow \quad S_{k-j}^{(j)}$$

and so, for $k > j$,

$$S_k^{(j)} = S_{k-1}^{(j)} + S_{k-2}^{(j)} + ... + S_{k-j}^{(j)}$$

This shows that $S_k^{(j)} = F_{k+2}^{(j)}$, as one can see comparing the following values of $S_k^{(j)}$ for small values of j with those values of $F_n^{(j)}$ shown in table (4).

$k = 0$	1	2	3	4	5	6
$S_k^{(2)} = 1$	2	3	5	8	13	21
$S_k^{(3)} = 1$	2	4	7	13	24	44
$S_k^{(4)} = 1$	2	4	8	15	29	56
$S_k^{(5)} = 1$	2	4	8	16	31	61
$S_k^{(6)} = 1$	2	4	8	16	32	63

3 The waiting time

3.1 The distribution

Let $T^{(j)}$ be the random variable determined by the waiting time for a run of j consecutive successes in a Bernoulli experiment, as defined in the preliminaries. The exact distribution of $T^{(j)}$ seems to be intractable, except for $p = 1/2$. In this case, every equal length sequence has the same probability and the distribution of $T^{(j)}$ is nicely given by:

$$
\mathrm{P}\left[T^{(j)}=n\right]=\begin{cases}0 & \text{for } n<j \quad (5)\\ \left(\dfrac{1}{2}\right)^{n} & \text{for } n=j \text{ or } n=j+1 \quad (6)\\ S_{n-(j+1)}^{(j)}\left(\dfrac{1}{2}\right)^{n} & \text{for } n>j+1 \quad (7)\end{cases}
$$

(5) is trivial, (6) is because the only allowed sequences are

$$\text{for } n=j \qquad \underbrace{1\ \ 1\ \ \ldots\ \ 1}_{j}$$

$$\text{for } n=j+1 \qquad \underbrace{0\ \ 1\ \ \ldots\ \ 1}_{j+1}$$

and (7) is because good sequence finishes in $\underbrace{0\ \ 1\ \ \ldots\ \ 1}_{j+1}$ preceded by sequences of size $n-(j+1)$ with no runs of j consecutive 1's. Remembering that $S_k^{(j)}=F_{k+2}^{(j)}$, if we let $k=n-j+1$, we can rewrite the distribution of $T^{(j)}$ as

$$\mathrm{P}\left[T^{(j)}=k+j-1\right]=F_k^{(j)}\left(\frac{1}{2}\right)^{k+j-1} \qquad (k=1,2,\ldots)$$

3.2 The expected value

As an introductory example, we derive the expected value for $T^{(2)}$. In this case, there is a very well known closed form for $F_k^{(2)}$ (Graham, R. L., Knuth, D. E., Patashnik, 1995):

$$F_k^{(2)}=F_k=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k-\left(\frac{1-\sqrt{5}}{2}\right)^k\right].$$

The distribution of $T^{(2)}$ is

$$\mathrm{P}\left[T^{(2)}=k+1\right]=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k-\left(\frac{1-\sqrt{5}}{2}\right)^k\right]\left(\frac{1}{2}\right)^{k+1} \quad \text{for } k\geq 1$$

and the expected value

$$
\begin{aligned}
\mathrm{E}[T^{(2)}] &= \sum_{j=2}^{\infty} j\ \mathrm{P}[T^{2} = j] \\
&= \sum_{j=2}^{\infty} j \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{j-1} - \left(\frac{1-\sqrt{5}}{2} \right)^{j-1} \right] \left(\frac{1}{2} \right)^{j} \\
&= \frac{1}{2\sqrt{5}} \sum_{j=2}^{\infty} j \left[\left(\frac{1+\sqrt{5}}{4} \right)^{j-1} - \left(\frac{1-\sqrt{5}}{4} \right)^{j-1} \right] \\
&= \frac{1}{2\sqrt{5}} \left[\left(\sum_{j=2}^{\infty} j x^{j-1} \right) - \left(\sum_{j=3}^{\infty} j y^{j-1} \right) \right]_{x=\frac{1+\sqrt{5}}{4},\ y=\frac{1-\sqrt{5}}{4}}
\end{aligned}
$$

$$
\sum_{j=2}^{\infty} j x^{j-1} = \sum_{j=2}^{\infty} \frac{d}{dx}\left(x^{j} \right) = \frac{d}{dx}\left(\sum_{j=2}^{\infty} x^{j} \right) = \frac{d}{dx}\left(\frac{x^{2}}{1-x} \right) = \frac{2x - x^{2}}{(1-x)^{2}}
$$

$$
\mathrm{E}[T^{2}] = \frac{1}{2\sqrt{5}} \left[\frac{2x - x^{2}}{(1-x)^{2}} - \frac{2y - y^{2}}{(1-y)^{2}} \right] = 6
$$

Taking the general case, the expected value of $T^{(j)}$ is given by

$$
\begin{aligned}
\mathrm{E}\left[T^{(j)} \right] &= \sum_{k=1}^{\infty} (k + j - 1) F_{k}^{(j)} \left(\frac{1}{2} \right)^{k+j-1} \\
&= \sum_{k=1}^{\infty} (k + j) F_{k}^{(j)} \left(\frac{1}{2} \right)^{k+j-1} - \sum_{k=1}^{\infty} F_{k}^{(j)} \left(\frac{1}{2} \right)^{k+j-1} \\
&= \sum_{k=1}^{\infty} \frac{d}{dx} \left[F_{k}^{(j)} (x)^{k+j} \right]_{x=\frac{1}{2}} - \left[(x)^{j-1} \sum_{k=1}^{\infty} F_{k}^{(j)} (x)^{k} \right]_{x=\frac{1}{2}} \\
&= \frac{d}{dx} \left[(x)^{j} \sum_{k=1}^{\infty} \left[F_{k}^{(j)} (x)^{k} \right] \right]_{x=\frac{1}{2}} - \left[(x)^{j-1} \sum_{k=1}^{\infty} F_{k}^{(j)} (x)^{k} \right]_{x=\frac{1}{2}} \\
&= \left[j (x)^{j-1} \sum_{k=1}^{\infty} \left[F_{k}^{(j)} (x)^{k} \right] \right]_{x=\frac{1}{2}} + \left[(x)^{j} \frac{d}{dx} \sum_{k=1}^{\infty} \left[F_{k}^{(j)} (x)^{k} \right] \right]_{x=\frac{1}{2}} + \\
&\qquad - \left[(x)^{j-1} \sum_{k=1}^{\infty} F_{k}^{(j)} (x)^{k} \right]_{x=\frac{1}{2}}
\end{aligned}
$$

We remember that

$$\sum_{n=1}^{\infty} x^n \; F_n^{(j)} = \frac{x}{1-x-x^2-\ldots-x^n}$$

and so

$$\left[\sum_{k=1}^{\infty} x^k \; F_k^{(j)}\right]_{x=\frac{1}{2}} = \frac{\frac{1}{2}}{1-\frac{1}{2}-\left(\frac{1}{2}\right)^2-\ldots-\left(\frac{1}{2}\right)^j} = \frac{\frac{1}{2}}{\left(\frac{1}{2}\right)^j} = 2^{j-1}$$

$$\frac{d}{dx}\left[\sum_{n=1}^{\infty} x^n \; F_n^{(j)}\right]_{x=\frac{1}{2}} = \frac{d}{dx}\left[\frac{x}{1-x-x^2-\ldots-x^j}\right]_{x=\frac{1}{2}}$$

$$= \left[\frac{\left(1-x-x^2-\ldots-x^j\right)+x(1+2x+\ldots+(j-1)x^{j-2}+jx^{j-1})}{\left(1-x-x^2-\ldots-x^j\right)^2}\right]_{x=\frac{1}{2}}$$

$$\left[x(1+2x+3x^2+\ldots+(j-1)x^{j-2}+jx^{j-1})\right]_{x=\frac{1}{2}} = \left[x\frac{d}{dx}\sum_{i=1}^{j} x^i\right]_{x=\frac{1}{2}}$$

$$= \left[x\frac{d}{dx}\frac{x\left(1-x^j\right)}{1-x}\right]_{x=\frac{1}{2}}$$

$$= \left[x\frac{(1-x)\left[\left(1-x^j\right)-jx^j\right]+x\left(1-x^j\right)}{(1-x)^2}\right]_{x=\frac{1}{2}}$$

$$= 2-\left(2+j\right)\left(\frac{1}{2}\right)^j$$

$$\frac{d}{dx}\left[\sum_{n=1}^{\infty} x^n \; F_n^{(j)}\right]_{x=\frac{1}{2}} = \frac{\left(\frac{1}{2}\right)^j+2-\left(2+j\right)\left(\frac{1}{2}\right)^j}{\left(\frac{1}{2}\right)^{2j}} = 2^j\left(2^{j+1}-1-j\right)$$

$$\mathrm{E}\left[T^{(j)}\right] = j\left(\frac{1}{2}\right)^{j-1} 2^{j-1} + \left(\frac{1}{2}\right)^{j} 2^{j}\left[2^{j+1} - 1 - j\right] - \left(\frac{1}{2}\right)^{j-1} 2^{j-1}$$

$$= 2(2^{j} - 1) \qquad (8)$$

Readers can check that (8) agrees with (1) for $p = \frac{1}{2}$.

3.3 The variance

$$\begin{aligned}
\mathrm{E}\left[\left(T^{(j)}\right)^2\right] &= \sum_{k=1}^{\infty} (k+j-1)^2 F_k^{(j)} \left(\frac{1}{2}\right)^{k+j-1} \\
&= \sum_{k=1}^{\infty} \left(k+j\right)(k+j-1) F_k^{(j)} \left(\frac{1}{2}\right)^{k+j-1} - \sum_{k=1}^{\infty} (k+j-1) F_k^{(j)} \left(\frac{1}{2}\right)^{k+j-1} \\
&= \frac{1}{2}\left[\frac{d^2}{dx^2} x^j \sum_{k=1}^{\infty} F_k^{(j)} x^k\right]_{x=\frac{1}{2}} - \mathrm{E}\left[T^{(j)}\right] \\
&= \frac{1}{2}\left[\frac{d^2}{dx^2}\left(\frac{x^{j+1}}{1 - x - x^2 - \ldots - x^j}\right)\right]_{x=\frac{1}{2}} - \mathrm{E}\left[T^{(j)}\right] \\
&= 2 + 2^{j+1}\left(2^{j+1} - 2j - 5\right)
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}\left[T^{(j)}\right] &= 2 + 2^{j+1}\left(2^{j+1} - 2j - 5\right) - 2(2^j - 1) - \left[2(2^j - 1)\right]^2 \\
&= 2^{j+1}\left(2^{j+1} - 2j - 1\right) - 2
\end{aligned}$$

We list some values of the mean and variance of $T^{(j)}$:

j	$\mathrm{E}\left[T^{(j)}\right]$	$\mathrm{Var}\left[T^{(j)}\right]$
2	6	22
3	14	142
4	30	734
5	62	3.390
6	126	14.718

Conclusions

Recurrence formulas for the expected value and variance of the waiting time for a run of n successes in a sequence of Bernoulli trials can be solved to a closed form for any value of the parameter p.

The j-step Fibonacci numbers allow the explicit calculation of the probability distribution function for waiting time for a run of n successes in a sequence of Bernoulli trials.

Obtained the closed form for the j-step Fibonacci numbers generator function, it was possible to get again the expected value and variance .

CHAVES, L. M.; SOUZA, D. J. de. Tempo de espera por *n* sucessos consecutivos em uma seqüência de ensaios de Bernoulli. *Rev. Bras. Biom.*, São Paulo, v.25, n.4, p.101-113, 2007.

- RESUMO: Usamos os números de Fibonacci de *j*-passos para obter a média e a variância da distribuição do tempo de espera por *n* sucessos consecutivos em uma seqüência de ensaios de Bernoulli. Para o caso do parâmetro $p = 1/2$ a distribuição exata foi calculada.
- PALAVRAS-CHAVE: Números de Fibonacci; tempo de espera; seqüência de Bernoulli.

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