SPATIAL CORRELATION OF FUNCTIONS DEFINED ON DISCS

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▪ *ABSTRACT: This article is a generalization of the concepts of spatial correlation, brought to a purely mathematical approach from the scope of Geostatistics, in the case in which we have two real-valued continuous functions defined on a topological closed disc with boundary locally flat of a Euclidean space.*

▪ *KEYWORDS: Closed disc with boundary locally flat; real-function defined on a disc; spatial dependence; spatial correlation; cross semivariogram.*

1 Introduction

Let $\mathbb{D} \subset \mathbb{R}^m$ be a closed m-disc in the m-dimensional Euclidean space with boundary $\partial\mathbb{D}$ locally flat. Let $f, g : \mathbb{D} \to \mathbb{R}$ be continuous real-valued functions. In this paper we study the so-called μ-spatial correlation (or μ-correlation) between f and g. This concept generalizes the Geostatistical concept called spatial correlation defined only when f and g are defined on discrete subsets of a 2-dimensional disc. We start by introducing the so-called cross μ-semivariogram $\gamma_{f,g}$ induced by f and g. Its definition is quite complicated and requires very precise mathematical reasoning. The main steps of the construction are in the proof that $\gamma_{f,g}$ is a continuous function and $\gamma_{f,g}(0) = 0$. In Geostatistics, we define $c_0 = \gamma_{f,g}(0)$ and this number is not zero, in general, being a kind of discontinuity of the sampling of f and g (see Goovaerts (1997)). In our context, f and g are continuous in whole disc, then it is reasonable to think that $\gamma_{f,g}(0) = 0$. After presenting these generalizations, we study the self-μ-correlation of a function f, that is, the μ-correlation of f with itself. We show that $\gamma_{f,f} = \gamma_f$ (the function γ_f is define by Fenille (2008) as a generalization of the semivariogram function) and that a continuous function is μ-correlated with itself if and only if it is μ-spatially dependent (the μ-spatial dependence is defined by Fenille (2008) as a generalization of the spatial dependence). We also prove

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several results presenting conditions so that two functions are μ-correlated. Finally, we present a very interesting example (Example 6.1) which demonstrates that the μ-correlation is not an equivalence relation on the set of the μ-spatially dependent functions. This result has important practical implications.

2 μ-Spatial correlation

Let $\mathbb{R}^m$ be the m-dimensional Euclidian space, $m \geq 2$, endowed with the Lebesgue measure μ. We say that a subset $\mathbb{D} \subset \mathbb{R}^m$ is a *region* if $\mathbb{D}$ is a closed m-disc with boundary locally flat, that is, for each point $x \in \partial\mathbb{D}$, there is an open m-ball $B_r^m(x)$ such that $\partial\mathbb{D} \cap B_r^m(x)$ is homeomorphic to an open $(m-1)$-ball. If E is a compact k-dimensional subset of $\mathbb{R}^m$, we write $v_k(E)$ to denote the k-dimensional Lebesgue measure (or volume) of E.

Throughout the text, $f, g : \mathbb{D} \to \mathbb{R}$ are real-valued continuous functions defined on a region $\mathbb{D} \subset \mathbb{R}^m$. By compactness, f and g are bounded. We write $\delta(\mathbb{D})$ to denote the diameter of the region $\mathbb{D}$ and consider $\mathcal{H} = [0, \delta(\mathbb{D})]$. For each $h \in \mathcal{H} \backslash \{0\}$ and each $x \in \mathbb{D}$, we define

$$\mathbb{S}_h(x) = \partial B_h(x) \cap \mathbb{D}.$$

Note that $\mathbb{S}_h(x)$ is a compact subset of the $(m-1)$-sphere with center at x and radius h contained in $\mathbb{R}^m$, except in the particular case in which, for convenience, we define $\mathbb{S}_0(x) = \{x\}$. Now, note that $\mathbb{S}_h(x)$ can be disconnected. However, by the definition of region, $\mathbb{S}_h(x)$ has only a finite number of connected components.

Let $\mathcal{P}_{\mathbb{S}}(\mathbb{D})$ be the subset of the parts of $\mathbb{D}$ formed by the sets $\mathbb{S}_h(x)$. We define the function $\mathbb{S} : \mathcal{H} \times \mathbb{D} \to \mathcal{P}_{\mathbb{S}}(\mathbb{D})$ to be $\mathbb{S}(h, x) = \mathbb{S}_h(x)$. By Fenille (2008), we can equip $\mathcal{P}_{\mathbb{S}}(\mathbb{D})$ with a "natural" topology that makes $\mathbb{S}$ a continuous map.

We also define the function *circular volume* $\mathcal{N} : \mathcal{H} \to \mathbb{R}$ of $\mathbb{D}$ by

$$\mathcal{N}(h) = \frac{1}{2} \int_{\mathbb{D}} v_{m-1}(\mathbb{S}_h(x))\, d\mu(x).$$

By Fenille (2008) this function is continuous, bounded and nonnegative. Moreover, by Proposition 1.1 proved by Fenille (2008) we have:

Proposition 2.1. $\mathcal{N}(h) = 0$ *if and only if* $h = 0$ *or* $h = \delta(\mathbb{D})$.

Let $f, g : \mathbb{D} \to \mathbb{R}$ be arbitrary continuous functions. We define the function $\mathcal{M}_{f,g} : \mathcal{H} \times \mathbb{D} \to \mathbb{R}$, called *local circular μ-correlation* of f and g, by

$$\mathcal{M}_{f,g}(h, x) = \int_{\mathbb{S}_h(x)} [f(x) - f(y)][g(x) - g(y)]\, d\mu(y).$$

Note that the value $\mathcal{M}_{f,g}(h, x)$ is well defined even if $\mathbb{S}_h(x)$ in disconnected. In fact, if $\mathbb{S}_h(x)$ has k connected components, say $\mathbb{S}_h^1(x), \cdots, \mathbb{S}_h^k(x)$, then

$$\mathcal{M}_{f,g}(h, x) = \sum_{i=1}^{k} \int_{\mathbb{S}_h^i(x)} [f(x) - f(y)][g(x) - g(y)]\, d\mu(y).$$

Since the function $\mathbb{S}$ is continuous and $\mathbb{S}_h(x)$ is a compact subset of $\mathbb{D}$ and, moreover, $\mathcal{H}$ and $\mathbb{D}$ are compact spaces, is trivial to prove the following:

Lemma 2.2. *The function $\mathcal{M}_{f,g}$ is continuous and bounded.*

Remark that, differently from the function $\mathcal{M}_f$ defined by Fenille (2008), $\mathcal{M}_{f,g}$ is not necessarily nonnegative.

Now, we study the function $\gamma^{\diamond}_{f,g} : \mathcal{H} \setminus \{0, \delta(\mathbb{D})\} \to \mathbb{R}$, unnamed for a while, given by

$$\gamma^{\diamond}_{f,g}(h) = \frac{1}{2\mathcal{N}(h)} \int_{\mathbb{D}} \mathcal{M}_{f,g}(h, x)\, d\mu(x).$$

Proposition 2.3. *The function $\gamma^{\diamond}_{f,g}$ is continuous.*

Proof: We know that the functions $\mathcal{N}$ and $\mathcal{M}_{f,g}$ are continuous and bounded. Moreover, by Proposition 2.1, $\mathcal{N}(h) \neq 0$ for all $h \in \mathcal{H} \setminus \{0, \delta(\mathbb{D})\}$. Hence, since $\mathbb{D}$ is compact, for any $h_0 \in \mathcal{H} \setminus \{0, \delta(\mathbb{D})\}$, we have

$$\lim_{h \to h_0} \gamma^{\diamond}_{f,g}(h) = \frac{1}{2\mathcal{N}(h_0)} \int_{\mathbb{D}} \mathcal{M}_{f,g}(h_0, x)\, d\mu(x) = \gamma^{\diamond}_{f,g}(h_0).$$

□

We would like to extend continuously $\gamma^{\diamond}_{f,g}$ to a function $\gamma_{f,g}$ defined also in the origin. To make this possible, it is necessary and sufficient that the limit $\lim_{h \to 0} \gamma^{\diamond}_{f,g}$ exists. Next, we present two results to show that this limit exists and is equal to zero.

Proposition 2.4. *Let $f, g : \mathbb{D} \to \mathbb{R}$ be two continuous function. For each $\varepsilon > 0$, let $g_\varepsilon : \mathbb{D} \to \mathbb{R}$ be a continuous ε-approximation of g. Then the nets $\{g_\varepsilon\}_\varepsilon$ and $\{\gamma^{\diamond}_{f,g_\varepsilon}\}_\varepsilon$ converge uniformly to g and $\gamma^{\diamond}_{f,g}$, respectively, when ε converges to zero.*

Proof: It follows directly from definition of ε-approximation and from the compactness of $\mathbb{D}$ that $\{g_\varepsilon\}_\varepsilon$ converges uniformly to g when $\varepsilon \to 0$. Now, by assumption, $|g(x) - g_\varepsilon(x)| \leq \varepsilon$ for all $x \in \mathbb{D}$. Thus, for any $x, y \in \mathbb{D}$, we have

$$g(x) - g(y) - 2\varepsilon \leq g_\varepsilon(x) - g_\varepsilon(y) \leq g(x) - g(y) + 2\varepsilon.$$

For each pair $(h, x) \in \mathcal{H} \times \mathbb{D}$, take the following subsets of $\mathbb{S}_h(x)$,

$${}_f\mathbb{S}^+_h(x) = \{y \in \mathbb{S}_h(x) : f(x) - f(y) > 0\},$$

$${}_f\mathbb{S}^-_h(x) = \{y \in \mathbb{S}_h(x) : f(x) - f(y) < 0\}.$$

For each $y \in {}_f\mathbb{S}^+_h(x)$, we have

$$\begin{aligned}[f(x) - f(y)][g(x) - g(y) - 2\varepsilon] &\leq [f(x) - f(y)][g_\varepsilon(x) - g_\varepsilon(y)] \\ &\leq [f(x) - f(y)][g(x) - g(y) + 2\varepsilon].\end{aligned}$$

On the other hand, for $y \in {}_f\mathbb{S}_h^-(x)$, we have

$$[f(x)-f(y)][g(x)-g(y)+2\varepsilon] \leq [f(x)-f(y)][g_\varepsilon(x)-g_\varepsilon(y)] \\ \leq [f(x)-f(y)][g(x)-g(y)-2\varepsilon].$$

Now, it is clear that, for $\varphi = g$ or g_ε, we have

$$\mathcal{M}_{f,\varphi}(h) = \int_{\mathbb{S}_h^+(x)} [f(x)-f(y)][\varphi(x)-\varphi(y)]d\mu(y) + \int_{\mathbb{S}_h^-(x)} [f(x)-f(y)][\varphi(x)-\varphi(y)]d\mu(y).$$

Thus, the latter two inequalities mean that

$$\mathcal{M}_{f,g}(h,x) - R_\varepsilon^f(h,x) \leq \mathcal{M}_{f,g_\varepsilon}(h,x) \leq \mathcal{M}_{f,g}(h,x) + R_\varepsilon^f(h,x), \tag{1}$$

where $R_\varepsilon^f : \mathcal{H} \times \mathbb{D} \to \mathbb{R}$ is the nonnegative continuous function

$$R_\varepsilon^f(h,x) = 2\varepsilon \int_{{}_f\mathbb{S}_h^+(x)} [f(x)-f(y)]d\mu(y) - 2\varepsilon \int_{{}_f\mathbb{S}_h^-(x)} [f(x)-f(y)]d\mu(y).$$

Now, note that the net of functions $\{R_\varepsilon^f\}_\varepsilon$ converges uniformly to the zero function when $\varepsilon \to 0$. Thus, from (1), we conclude that the net $\{\mathcal{M}_{f,g_\varepsilon}\}_\varepsilon$ converges uniformly to the function $\mathcal{M}_{f,g}$ when $\varepsilon \to 0$.

Finally, it is not difficult to check that $\{\gamma_{f,g_\varepsilon}\}_\varepsilon$ converges uniformly to $\gamma_{f,g}$ when $\varepsilon \to 0$. This concludes the proof. □

Theorem 2.5. *Let $f, g : \mathbb{D} \to \mathbb{R}$ be continuous functions. Then* $\lim_{h\to 0} \gamma_{f,g}^\diamond = 0$.

Proof: By Folland (1999, Proposition 8.17, p.245), given $\varepsilon > 0$, there is an infinitely differentiable function $g_\varepsilon : \mathbb{D} \to \mathbb{R}$ which we can take as ε-approximation of g. By Proposition 2.4, the net $\{\gamma_{f,g_\varepsilon}^\diamond\}_\varepsilon$ converges uniformly to $\gamma_{f,g}^\diamond$ when ε converges to zero. Thus, it is sufficient to prove that $\lim_{h\to 0} \gamma_{f,g_\varepsilon}^\diamond = 0$ for all ε.

Initially, note that, since f is continuous and $\mathbb{D}$ is compact, there is a real number $k_f > 0$ such that $|f(x) - f(y)| \leq k_f$ for any $x, y \in \mathbb{D}$.

Now, since each $g_\varepsilon : \mathbb{D} \to \mathbb{R}$ is a $\mathcal{C}^\infty$-function defined on a closed subset of $\mathbb{R}^m$, the Tietze's Theorem (see Lima (1972, p.248)) shows that g_ε can be extended to a $\mathcal{C}^\infty$-function $\widetilde{g}_\varepsilon : \mathbb{R}^m \to \mathbb{R}$. By the Mean Value Inequality (see Lima (2004, p.35)), for all $y \in \mathbb{S}_h(x)$, we have $|g_\varepsilon(x) - g_\varepsilon(y)| \leq h \sup\{|g'_\varepsilon(z)| : z \in [x,y]\}$. Then, by compactness, there is a real number $k_\varepsilon > 0$ such that $|g_\varepsilon(x) - g_\varepsilon(y)| \leq hk_\varepsilon$, for all $y \in \mathbb{S}_h(x)$. This means that, for any pair $(h,x) \in \mathcal{H} \times \mathbb{D}$, we have

$$|M_{f,g_\varepsilon}(h,x)| \leq \int_{\mathbb{S}_h(x)} |f(x)-f(y)||g_\varepsilon(x)-g_\varepsilon(y)|d\mu(y) \leq k_f k_\varepsilon h\, v_{m-1}(\mathbb{S}_h(x)).$$

Now, since $\mathbb{S}_h(x)$ is a subset of the boundary $\partial B_h^m(x)$ of the m-dimensional ball with center at x and radius h, it follows that $v_{m-1}(\mathbb{S}_h(x)) \leq v_{m-1}(\partial B_h^m(0)) =$

$l_m h^{m-1}$, where l_m is a positive real number which depends exclusively on dimension m. Therefore, the above inequality means that

$$|\mathcal{M}_{f,g_\varepsilon}(h,x)| \leq k_f k_\varepsilon\, l_m h^m.$$

Integrating on $\mathbb{D}$ with respect to $\mu(x)$, we obtain the inequality

$$\left|\int_{\mathbb{D}} \mathcal{M}_{f,g_\varepsilon}(h,x) d\mu(x)\right| \leq \int_{\mathbb{D}} |\mathcal{M}_{f,g}(h,x)| \mu(x) \leq k_f k_\varepsilon\, l_m h^m\, v_m(\mathbb{D}).$$

On the other hand, a similar argument shows that, for all $h \in \mathcal{H} \setminus \{0, \delta(\mathbb{D})\}$,

$$0 < \mathcal{N}(h) \leq \frac{1}{2}\, v_m(\mathbb{D})\, l_m h^{m-1}.$$

From these latter two inequalities, we conclude that, for all $h \in \mathcal{H} \setminus \{0, \delta(\mathbb{D})\}$,

$$|\gamma^\diamond_{f,g_\varepsilon}(h)| = \frac{\left|\int_{\mathbb{D}} \mathcal{M}_{f,g_\varepsilon}(h,x) d\mu(x)\right|}{2\, \mathcal{N}(h)} \leq k_f k_\varepsilon h.$$

Finally, $\lim_{h\to 0} \gamma^\diamond_{f,g_\varepsilon}(h) = 0$? □

Henceforth, we denote $\mathcal{H}_{\mathbb{D}} = \mathcal{H} \setminus \{\delta(\mathbb{D})\}$. By the latter theorem, we can always extend the function $\gamma^\diamond_{f,g}(h)$ continuously on the domain $\mathcal{H}_{\mathbb{D}}$ in the following way:

Definition 2.6. We call *cross μ-semivariogram* induced by f and g the function $\gamma_{f,g} : \mathcal{H}_{\mathbb{D}} \to \mathbb{R}$ given by $\gamma_{f,g}(h) = \gamma^\diamond_{f,g}(h)$ if $h \neq 0$ and $\gamma_{f,g}(0) = 0$.

From Theorem 2.5 and Definition 2.6 we have the following result:

Theorem 2.7. *The function $\gamma_{f,g}$ is continuous.*

Definition 2.8. We say that two continuous functions $f, g : \mathbb{D} \to \mathbb{R}$ are *μ-spatially correlated* (or *μ-correlated*) if there is a real number a, with $0 < a \leq \delta(\mathbb{D})$, such that $\gamma_{f,g}$ is monotonous non-constant on $[0, a)$. The supreme of the real numbers with this property is called the *range* of the μ-spatial correlation (or *μ-correlation*) between f and g and it is denoted by $\mathfrak{a}$. Furthermore, if f and g are μ-spatially correlated, then we say that

- f and g are *directly μ-correlated* if $\gamma_{f,g}$ is strictly increasing on $[0, \mathfrak{a})$.
- f and g are *inversely μ-correlated* if $\gamma_{f,g}$ is strictly decreasing on $[0, \mathfrak{a})$.

Example 2.9. Let $f : \mathbb{D} \to \mathbb{R}$ be a continuous function. If f is almost always constant, then f is constant and there is not a continuous function $g : \mathbb{D} \to \mathbb{R}$ which is μ-correlated with f, since in this case $\mathcal{M}_{f,g}(h,x) = 0$ for every pair $(h,x) \in \mathcal{H}\times\mathbb{D}$, and so $\gamma_{f,g} : \mathcal{H}_{\mathbb{D}} \to \mathbb{R}$ is the zero function. □

Physical interpretation: Two μ-spatially dependent functions $f, g : \mathbb{D} \to \mathbb{R}$ are μ-correlated if they present similar variation on the region $\mathbb{D}$. More precisely, f and g are directly μ-correlated if, "in general", they present increase and decrease on the same parts or directions, that is, if f increases (decreases), so g does. On the other hand, f and g are inversely μ-correlated if, "in general", they present inverted increase and decrease in same parts or directions, that is, if f increases (decreases), then g decreases (increases). Here, "in general" means that such a behavior of the functions need not occur throughout the region, but in some subset and with an intensity great enough to compensate for a possible different behavior in another part of the region.

3 Self-μ-correlation versus μ-spatial dependence

In this section, we study the existence of μ-correlation between two identical functions, that is, the μ-correlation of a function with itself. Although, at first, it clearly seems that any continuous function $f : \mathbb{D} \to \mathbb{R}$ is directly μ-correlated with itself, this is not true, in general. In fact, by the latter example, any constant function is not μ-correlated with itself. We will prove that a continuous function f is μ-correlated with itself if and only if it is μ-spatially dependent (see Definition 1.2 of Fenille (2008)), and in this case, f is directly μ-correlated with itself. áce-.1cm Let $f : \mathbb{D} \to \mathbb{R}$ be a continuous function. For each $(h, x) \in \mathcal{H} \times \mathbb{D}$, we have

$$\mathcal{M}_{f,f}(h,x) = \int_{\mathbb{S}_h(x)} |f(x) - f(y)|^2 d\mu(y) = \mathcal{M}_f(h,x),$$

where $\mathcal{M}_f$ is the local circular μ-variance defined by Fenille (2008). It follows that

$$\gamma_{f,f}(h) = \gamma_f(h), \ for \ all \ h \in \mathcal{H}_{\mathbb{D}}, \tag{2}$$

where γ_f is the μ-semivariogram induced by f, defined by Fenille (2008).

Sentence (2), Theorem 2.5 and Definition 2.6 together are sufficient to guarantee the next theorem, which improves Proposition 1.3 of Fenille (2008) and can be used to rewrite more accurately the definition of the μ-semivariogram induced by a continuous function $f : \mathbb{D} \to \mathbb{R}$ (see Definition 1.1 of Fenille (2008)).

Theorem 3.1. *If $f : \mathbb{D} \to \mathbb{R}$ is a continuous function then $\gamma_f(0) = 0$.*

Theorem 3.2. *A continuous function $f : \mathbb{D} \to \mathbb{R}$ is μ-correlated with itself if and only if it is μ-spatially dependent. Moreover, if one of this sentences is true, then f is directly μ-correlated with itself.*

Proof: Let $f : \mathbb{D} \to \mathbb{R}$ be a continuous map. Then f is μ-correlated with itself if and only if the function $\gamma_{f,f} : \mathcal{H}_{\mathbb{D}} \to \mathbb{R}$ is monotone non-constant in some interval $[0, a) \subset \mathcal{H}_{\mathbb{D}}$. Thus, by sentence (2) and Theorem 1.1 of Fenille (2008), f is μ-correlated with itself if and only if the μ-semivariogram γ_f is strictly increasing on some interval $[0, a) \subset \mathcal{H}_{\mathbb{D}}$. Now, the latter fact occurs if and only if f is μ-spatially dependent. Furthermore, if γ_f is strictly increasing on $[0, a)$, then so $\gamma_{f,f}$ is and, finally, f is directly μ-correlated with itself. □

Physical Interpretation: A continuous function $f : \mathbb{D} \to \mathbb{R}$ is μ-spatially dependent if, in general, it presents similar values on close points and presents discrepant values on distant points.

4 Some results on μ-correlation

In this section, we prove some results on existence of the μ-correlation, which brings strong evidences of the physical interpretation of the μ-correlation.

Proposition 4.1. *Let $f : \mathbb{D} \to \mathbb{R}$ be a μ-spatially dependent function. Let $g : \mathbb{D} \to \mathbb{R}$ be the function $g(x) = \alpha f(x) + \beta$, for all $x \in \mathbb{D}$, where $\alpha, \beta \in \mathbb{R}$.*

1. *If $\alpha = 0$, then f and g are not μ-correlated;*
2. *If $\alpha > 0$, then f and g are directly μ-correlated;*
3. *If $\alpha < 0$, then f and g are inversely μ-correlated.*

Proof: For each pair $(h, x) \in \mathcal{H} \times \mathbb{D}$, we have

$$\begin{aligned}\mathcal{M}_{f,g}(h,x) &= \int_{\mathbb{S}_h(x)} [f(x) - f(y)][\alpha f(x) + \beta - \alpha f(y) - \beta] d\mu(y) \\ &= \alpha \int_{\mathbb{S}_h(x)} |f(x) - f(y)|^2 d\mu(y) = \alpha\, \mathcal{M}_f(h,x),\end{aligned}$$

where $\mathcal{M}_f$ is the local circular μ-variance defined by Fenille (2008). Thus, for each $h \in \mathcal{H}_{\mathbb{D}}$, $h \neq 0$, we have

$$\gamma_{f,g}(h) = \frac{\alpha}{2\mathcal{N}(h)} \int_{\mathbb{D}} \mathcal{M}_f(h,x) d\mu(x) = \alpha\, \gamma_f(h).$$

Moreover, since $\gamma_{f,g}(0) = 0 = \gamma_f(0)$, it follows that $\gamma_{f,g}(h) = \alpha\, \gamma_f(h)$ for all $h \in \mathcal{H}_{\mathbb{D}}$. Now, since f is μ-spatially dependent, the μ-semivariogram γ_f is strictly increasing on an interval $[0, \mathfrak{a}) \subset \mathcal{H}_{\mathbb{D}}$. The result follows by analyzing the behavior of $\gamma_{f,g} = \alpha\gamma_f$ in terms of α. □

Proposition 4.2. *Let $f, g : \mathbb{D} \to \mathbb{R}$ be continuous functions. Then f is directly (inversely) μ-correlated with g if and only if f is inversely (directly) μ-correlated with $-g$.*

Proof: It is sufficient to note that $\mathcal{M}_{f,-g} = -M_{f,g}$ and so $\gamma_{f,-g} = -\gamma_{f,g}$. Thus, if $\gamma_{f,g}$ is increasing on $[0, \mathfrak{a})$, then $\gamma_{f,-g}$ is decreasing on $[0, \mathfrak{a})$, and vice versa. □

The next result is an immediately consequence of Proposition 2.4

Proposition 4.3. *Let $f, g : \mathbb{D} \to \mathbb{R}$ be directly (inversely) μ-correlated functions. If g_ε is a ε-approximation of g, then f and g_ε are directly (inversely) μ-correlated, for ε small enough.*

Corollary 4.4. *Let $f : \mathbb{D} \to \mathbb{R}$ be a μ-spatially dependent function. If $f_\varepsilon : \mathbb{D} \to \mathbb{R}$ is a ε-approximation of f, then f and f_ε are directly μ-correlated, for ε small enough.*

Proof: Since f is μ-spatially dependent, Theorem 3.2 means that f is directly μ-correlated with itself. Then, the proof is a consequence of Proposition 4.3. □

Proposition 4.5. *Let $f : \mathbb{D} \to \mathbb{R}$ be a μ-spatially dependent function. Then every ε-approximation of f, for ε small enough, is μ-spatially dependent.*

Proof: It follows from Proposition 2.4 and sentence (2) that the net of μ-semivariograms $\{\gamma_{f_\varepsilon}\}_\varepsilon$ converges uniformly to γ_f when $\varepsilon \to 0$. □

The following lemma is an exercise of Real Analysis, see Lima (2006, p.252).

Lemma 4.6. *Let $\Phi : [a, b] \to \mathbb{R}$ be a continuous function and $\varepsilon > 0$ a real number. Then Φ has a polygonal ε-approximation $\tilde{\Phi} : [a, b] \to \mathbb{R}$. Furthermore, if f is strictly increasing (decreasing), then the function $\tilde{\Phi}$ can be chosen strictly increasing (decreasing).*

Theorem 4.7. *Let $f : \mathbb{D} \to \mathbb{R}$ be a μ-spatial dependent function and $\Phi : [a, b] \to \mathbb{R}$ be a strictly increasing (decreasing) continuous function with $[a, b] \supset im(f)$. Then f and $\Phi \circ f$ are directly (inversely) μ-correlated.*

Proof: We assume that Φ is strictly increasing (the other case is similar). By the previous lemma, for every real number $\varepsilon > 0$, Φ can be ε-approximated by a strictly increasing polygonal $\tilde{\Phi} : [a, b] \to \mathbb{R}$, that is, there are a partition $a = t_1 < t_2 < \cdots < t_r = b$ of the interval $[a, b]$ and real numbers $\alpha_1, \beta_1 \ldots, \alpha_r, \beta_r$ such that, for each $t \in [t_i, t_{i+1}] \subset [a, b]$, the function $\tilde{\Phi}(t) = \alpha_i t + \beta_i$ satisfies $|\tilde{\Phi}(t) - \Phi(t)| < \varepsilon$ and, moreover, the $\alpha_i' s$ can be chosen positives.

Let $\tilde{f} = \tilde{\Phi} \circ f : \mathbb{D} \to \mathbb{R}$. Then, for each $x \in \mathbb{D}$ with $f(x) \in [t_i, t_{i+1}]$, we have $\tilde{f}(x) = \alpha_i f(x) + \beta_i$. Now, with several arguments as in the proof of Theorem 4.1, we can prove that f and $\tilde{f}$ are directly μ-correlated. On the other hand, $\tilde{f}$ is a ε-approximation of $\Phi \circ f$. Therefore, by Proposition 4.3, it follows that f and $\Phi \circ f$ are directly μ-correlated. □

This theorem is a stronger evidence of the physical interpretation of the μ-correlation.

5 The 1-dimensional special case

Let $\mathbb{D} \subset \mathbb{R}$ be a region. Then, $\mathbb{D}$ can not be anything other than a closed interval $[a, b]$ with $a < b$. In this case, $\mathcal{H} = [0, b-a]$ and, for every pair $(h, x) \in \mathcal{H} \times \mathbb{D}$, we have

$$\mathbb{S}_h(x) = \begin{cases} \emptyset & \text{if} \quad x \notin [a+h, b-h]; \\ \{x+h\} & \text{if} \quad x \in [a, a+h) \setminus (b-h, b]; \\ \{x-h\} & \text{if} \quad x \in (b-h, b] \setminus [a, a+h); \\ \{x+h, x-h\} & \text{if} \quad x \in [a+h, b-h]. \end{cases}$$

By Fenille (2008), $\mathcal{N}(h) = b - a - h$ for all $h \in \mathcal{H}$. Let $f, g : [a, b] \to \mathbb{R}$ be continuous function. Then

$$\mathcal{M}_{f,g}(h, x) = \sum_{y \in \mathbb{S}_h(x)} [f(x) - f(y)][g(x) - g(y)].$$

Thus, the function $\gamma_{f,g} : \mathcal{H}_{\mathbb{D}} \to \mathbb{R}$ is given by

$$\gamma_{f,g}(h) = \frac{1}{2(b-a-h)} \int_a^b \mathcal{M}_{f,g}(h, x) d\mu(x).$$

More precisely, if we write $f(x, \pm h) = f(x) - f(x \pm h)$ and $g(x, \pm h) = g(x) - g(x \pm h)$, then for each $h \in [0, (b-a)/2]$, we have

$$\gamma_{f,g}(h) = \frac{1}{2(b-a-h)} \left[\int_a^{a+h} f(x, +h) g(x, +h) dx + \int_{a+h}^{b-h} f(x, -h) g(x, -h) dx \right.$$

$$\left. + \int_{a+h}^{b-h} f(x, +h) g(x, +h) dx + \int_{b-h}^{b} f(x, -h) g(x, -h) dx \right].$$

So, for each $h \in ((b-a)/2, b-a)$, we have

$$\gamma_{f,g}(h) = \frac{1}{2(b-a-h)} \left[\int_a^{b-h} f(x, +h) g(x, +h) dx + \int_{a+h}^{b} f(x, -h) g(x, -h) dx \right].$$

If we are not interested in determining the range, but only detect the μ-correlation, then it is sufficient to determine $\gamma_{f,g}(h)$ for $h \in [0, (b-a)/2]$. The restriction of $\gamma_{f,g}$ on the subset $\mathcal{H}^* = [0, (b-a)/2] \subset \mathcal{H}_{\mathbb{D}}$ is called the *principal restriction* of the cross μ-semivariogram $\gamma_{f,g}$ and denoted by $\gamma^*_{f,g}$. In the next example, we determine only the principal restrictions of the involved croos μ-semivariograms.

6 An example and an important conclusion

Now, we present an example to show that the μ-correlation is not an equivalence relation on the set of the μ-spatially dependent functions.

Example 6.1. Let $f : [-1, 1] \to \mathbb{R}$ be the injection $f(x) = x$, let $g : [-1, 1] \to \mathbb{R}$ be the polynomial function $g(x) = x^2$ and let $\varphi : [-1, 1] \to \mathbb{R}$ be the exponential function $\varphi(x) = e^{x-1}$. We will prove that: **(i)** f, g and φ are μ-spatially dependents functions; **(ii)** f and g are not μ-correlated; **(iii)** f and φ are directly μ-correlated; **(iv)** φ and g are directly μ-correlated. Let's prove it: By the formula for $\gamma_{f,g}$ in the 1-dimensional case and sentence (2), it follows that:

Since $f(x) - f(x \pm h) = \mp h$, we have $[f(x) - f(x \pm h)]^2 = h^2$ and

$$\gamma_f^*(h) = \frac{1}{2(2-h)} \left[\int_{-1}^{-1+h} h^2 dx + \int_{-1+h}^{1-h} 2h^2 dx + \int_{1-h}^{1} h^2 dx \right] = h^2.$$

Now, since $g(x) - g(x \pm h) = -h^2 \mp 2xh$, a similar argument shows that

$$\gamma_g^*(h) = \frac{h^5 - 12h^4 + 12h^3}{6 - 3h}.$$

Again, since $\varphi(x) - \varphi(x \pm h) = e^{x-1} - e^{x \pm h - 1}$, we have

$$\gamma_\varphi^*(h) = \frac{(e^h - 1)^2 (e^{-4} - e^{-2h})}{2h - 4}.$$

The calculations can be manually done or with a help of a computer program with basic tools of the Integral Calculus. By an easy argument of the differential calculus, we prove that γ_f^* and γ_φ^* are strictly increasing on $\mathcal{H}^* = [0, 1]$ and γ_g^* is strictly increasing on $[0, 0.86] \subset \mathcal{H}^*$. Therefore, by Definition 1.2 of Fenille (2008), the functions f, g and φ are μ-spatially dependents.

Now, according to (**ii**), we will show that f and g are not μ-correlated. For this, remark that

$$\gamma_{f,g}^*(h) = \frac{1}{4 - 2h} \left[\int_{-1}^{-1+h} -h(h^2 + 2xh) dx + \int_{-1+h}^{1-h} -4xh^2 dx + \int_{1-h}^{1} h(h^2 - 2xh) \right] = 0.$$

Thus, the function $\gamma_{f,g}^*$ is constant on $\mathcal{H}^*$ and, therefore, f and g are not μ-correlated.

Now, the reader knows how to do the necessary calculations to check that

$$\gamma_{f,\varphi}^*(h) = \frac{h(1 - e^{2h})(e^{-2} - e^{-2h})}{2 - h},$$

and that this function is strictly increasing on $[0, 0.72] \subset \mathcal{H}^*$. Therefore, f and φ are directly μ-correlated, as stated in **(iii)**.

Furthermore, it is easy to check that

$$\gamma_{g,\varphi}^*(h) = \frac{e^{-2h}(e^h - 1)(3 + 2e^h + 3e^{2h})h^3}{4 - 2h}.$$

In order to check that this function is monotonous increasing on a neighborhood in the right of the origin, note that $\lim_{h\to 0} 1/\gamma^*_{g,\varphi} = +\infty$. Indeed, $\gamma^*_{g,\varphi}$ is strictly increasing on $\mathcal{H}^* = [0, 1]$. Therefore, g and φ are directly μ-correlated, in according to **(iv)**. □

Example 6.1 is very important to show that the μ-correlation is not an equivalent relation. We state this more precisely in the next theorem.

Theorem 6.2. *The μ-correlation is not an equivalence relation on the set of the μ-spatially dependent functions.*

Proof: In Example 6.1, the functions f is directly μ-correlated with φ and φ is directly μ-correlated with g. However, f is not μ-correlated φ. Therefore, although the μ-correlation checks the *reflexive* and *symmetric* properties on the set of the μ-spatially dependent functions, it does not check the *transitive* property. Thus, the μ-correlation is not an equivalence relation. □

Example 6.3. Let $\mathbb{D}$ be the region corresponding to São Paulo state and let $\mathcal{A}, \mathcal{P}, \mathcal{T} : \mathbb{D} \to \mathbb{R}$ be the "continuous" functions $\mathcal{A} =$*altitude*, $\mathcal{P} =$ *rainfall* e $\mathcal{T} =$ *air temperature*. By Fenille and Cardim (2007a, 2007b), this three functions are spatially dependent. Now, by Fenille and Cardim (2007a), $\mathcal{A}$ and $\mathcal{P}$ are directly correlated and by Fenille and Cardim (2007b), $\mathcal{A}$ and $\mathcal{T}$ are inversely correlated. By Theorem 6.2, this does not mean, immediately, that $\mathcal{P}$ and $\mathcal{T}$ are correlated phenomena; we need a specific analysis for this. □

7 An equivalence with geostatistics

In this section, we will explain in detail how to give the generalization of the geostatistical concept spatial correlation to the new concept μ-spatial correlation (or μ-correlation). We start by redefining the spatial correlation, bringing this concept to approach developed in this paper. This can be found in the scope of the Geostatistics in Goovaerts (1997).

Let $\mathbb{X}$ be a finite subset of $\mathbb{R}^m$ and let $f, g : \mathbb{D} \to \mathbb{R}$ be continuous functions defined in a region $\mathbb{D}$ of $\mathbb{R}^m$ containing $\mathbb{X}$. However, suppose the values of f and g are known only in the points of $\mathbb{X}$. Let d be the usual metric on the Euclidean space $\mathbb{R}^m$. We define the set of feasible distance of $\mathbb{X}$ to be

$$\mathcal{H}_{\mathbb{X}} = \{d(x, y) : x, y \in \mathbb{X},\ x \neq y\}.$$

For each feasible distance $h \in \mathcal{H}_{\mathbb{X}}$ and each point $x \in \mathbb{X}$, we define the set $\mathbb{S}_h[x] = \partial B_h(x) \cap \mathbb{X}$. Furthermore, for each $h \in \mathcal{H}_{\mathbb{X}}$, we define

$$\mathbb{X}[h] = \{x \in \mathbb{X} : \exists\ y \in \mathbb{X} \text{ satisfying } d(x, y) = h\}.$$

Now, we define the same function of Section 2, but changing $\mathbb{D}$ to $\mathbb{X}$, $\mathcal{H}$ to $\mathcal{H}_{\mathbb{X}}$ and the Lebesgue measure to the counting measure.

We start by defining the function $\mathcal{N}_0 : \mathcal{H}_{\mathbb{X}} \to \mathbb{R}$ by

$$\mathcal{N}_0(h) = \lceil \frac{1}{2} \sum_{x \in \mathbb{X}[h]} \#\mathbb{S}_h[x] \rceil,$$

where $\lceil \alpha \rceil$ denotes the largest integer less than or equal to α and $\#$ denotes cardinality. In practice, for each $h \in \mathcal{H}_{\mathbb{X}}$, the value $\mathcal{N}_0(h)$ is the quantity of pairs of points in $\mathbb{X}$ whose distance is h.

For each $x \in \mathbb{X}$, we denote $\mathcal{H}_{\mathbb{X}}[x] = \{d(x, y) : y \in \mathbb{X}\}$. Now, we define the function $\mathcal{M}^0_{f,g}[x] : \mathcal{H}_{\mathbb{X}}[x] \to \mathbb{R}$ by

$$\mathcal{M}^0_{f,g}[x](h) = \sum_{y \in \mathbb{S}_h[x]} [f(x) - f(y)][g(x) - g(y)].$$

Finally, we define the function $\gamma^0_{f,g} : \mathcal{H}_{\mathbb{X}} \to \mathbb{R}$ by

$$\gamma^0_{f,g}(h) = \frac{1}{2\mathcal{N}_0(h)} \sum_{x \in \mathbb{X}[h]} \mathcal{M}^0_{f,g}[x](h).$$

The reader can verify that, in the context of regionalized variables, this is the function called *cross semivariogram.* However, to agree with Definition 2.6, we call it *cross μ_0-semivariogram induced by f and g.* To maintain the compatibility with Geostatistics, we define:

Definition 7.1. We say that f and g are *μ_0-correlated* if there is a feasible distance $h_a \in \mathcal{H}_{\mathbb{X}}$, with $h_a > \min\mathcal{H}_{\mathbb{X}}$, such that $\gamma^0_{f,g}$ is monotonous non-constant on the set $[\min\mathcal{H}_{\mathbb{X}}, h_a] \cap \mathcal{H}_{\mathbb{X}}$.

All this shows that the geostatistical study of the spatial correlation between two regionalized variables is a special case of the study of the μ-correlation of two continuous functions defined on regions of the m-dimensional Euclidean space. The main particularity is, in fact, the change of the Lebesgue measure to the counting measure. The practical context of Geostatistics is, in general, even more restricted, since its main applications are about the study of natural phenomena on Earth's surface and therefore in dimension two.

Acknowledgments

I would like to express my thanks to Mario Henrique de Castro for his careful reading and helpful suggestions.

FENILLE, M. C. Correlação espacial de funções definidas em discos. *Rev. Bras. Biom.,* São Paulo, v.28, n.2, p.62-74, 2010.

▪ RESUMO: Este artigo apresenta uma generalização do conceito de correlação espacial, trazido do escopo da Geoestatística para uma abordagem puramente matemática, nos casos em que tem-se duas funções reais definidas em um disco fechado topológico com bordo localmente plano de algum espaço Euclideano.

▪ PALAVRAS-CHAVE: Disco fechado com bordo localmente plano; função real definida em um disco; dependência espacial; correlação espacial; semivariograma cruzado.

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Received in 4.01.2010.

Approved after revised in 11.05.2010.