CONTROLLING THE OSCILLATIONS OF A VARIABLE LENGTH PENDULUM

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▪ *ABSTRACT: An efficient method for stopping a pendulum's planar oscillations by variations in the pendulum's length is found. This strategy is accomplished by casting the problem as an optimal control problem. The pendulum's governing equations are deduced and using these equations the oscillation energy of the pendulum is found. The problem becomes a variational problem with constraints in which a functional which represent the oscillation energy of the pendulum is to be minimized. Using Pontryagin's Principle, optimal solutions are found. Finally, the effectiveness of the found strategies is illustrated graphically; analytical and numerical comparisons are made.*

▪ *KEYWORDS: Pendulum of variable length; oscillation energy; Pontryagin principle.*

1 Introduction

Routinely, machines (such as clocks, robot arms and cranes) use pendulum systems to operate. Sometimes cranes need to stop or reduce their oscillations when they are working. Previous strategies for stopping oscillations consist in moving the crane or moving the pivot point of its cord, but these strategies are inefficient and sometimes impossible to implement. The control of the angular oscillations of a variable length pendulum has been treated from different points of view.

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Heuristic experiments and simulations use different forms to stop or reduce pendulum oscillations by changing its cord's length. These strategies work by enlarging its length when the mass reaches its maximum speed and reducing its length when it reaches its minimum speed. This heuristic strategy is based on the following equation, which is called the control equation of a pendulum with variable length moving in the plane:

$$l\theta'' + 2l'\theta' + g\sin(\theta) = 0.$$

Note that the presence of the factor l' acts like a coefficient of friction when $l' > 0$.

There is abundant work relating to the control of the variable-length pendulum. A first approach to the problem using lagrangian dynamics is taken by Bressand and Rampazzo (1993) and Sira (1999). In Stilling and Szyszkowski (2002), this phenomenon is explained using the Coriolis inertia force and examining the energy variation during an oscillation cycle. Simple rules relating the sliding motions to the angular oscillation are proposed and assessed using numerical simulations. Rotational motions of the system can be decreased or increased by generating the Coriolis force that opposes or aids the motion by having the pendulum length increased or decreased respectively. In Akulenko and Nesterov and Akulenko (2009), periodic modes of small pendulum's oscillations are constructed to determine the limits of the regions of stability and instability of the lower position of equilibrium. The phenomenon is explained using the solution of periodic boundary value problems of eigenvalues and eigenfunctions, that is Sturm-Liouville type problems.

In this paper we introduce the equations that control the behavior of a pendulum with variable length in the plane. The equations are in terms of the length and oscillation angle which are functions of time, and can be found using Newton's second law. In the process of solving these equations, we find an equation which relates the sum of the kinetic and potential energy of the system. This is called the oscillation energy of the pendulum. In order to reduce the energy of the pendulum, we need to minimize the functional that represents this oscillation energy.

We first tried to solve the problem finding the extremals of the variational problem, we found the Euler Lagrange equations and got a nonlinear system of equations of second order, which was difficult to solve.

Pontryagin's principle is a result in optimal control which allows the control to be a discontinuous bounded function. This principle allows us to find extremals for specific types of functionals. We use this principle to find the minimum of the functional which represent the oscillatory energy of the pendulum.

Finally, we compare numerically the heuristic strategy and the proposed optimal strategy. We use the fourth order Runge Kutta method (see Suli and Mayers (2003)) to numerically solve the equation that controls the pendulum in the plane.

This article is organized as follows. In section 2, analysis of the problem and related theory used to solve the pendulum problem is shown. In section 3, Pontryagin's Principle and the solution to the pendulum problem are shown. Numerical results are given in section 4. Finally in section 5, the conclusions and future work related to this problem are presented.

2 Theory related to the pendulum system

2.1 Problem analysis

Figure 1 displays the scheme of a pendulum system in two dimensions with a variable length.

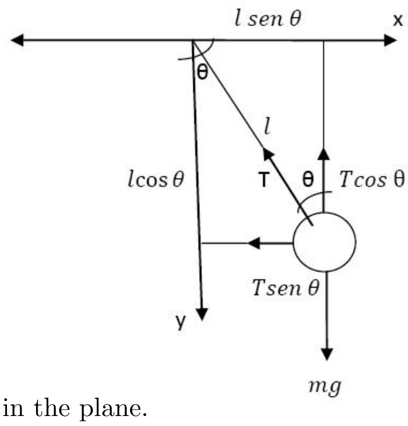

Figure 1 - Pendulum in the plane.

The equations related to this system can be found using Newton's second law, $F = ma$, where m is the pendulum's weight, a is the acceleration. In vector form, we have:

$$m(0, g) - (Tsin(\theta), Tcos(\theta)) = m(lsin(\theta), lcos(\theta))''$$

where l, θ are functions of time and T is the cord's tension. Taking time derivatives and simplifying, we have the system:

$$l''sin(\theta) + 2l'cos(\theta)\theta' + lcos(\theta)\theta'' - lsin(\theta)(\theta')^2 = -Tsin(\theta) \quad (1)$$

$$l''cos(\theta) - 2l'sen(\theta)\theta' - lcos(\theta)(\theta')^2 - lsin(\theta)\theta'' = g - Tcos(\theta) \quad (2)$$

These equations control the pendulum with variable length in the plane.
By multiplying (1) and (2), respectively, by $cos(\theta)$, $sin(\theta)$ and adding them, we obtain the equation:

$$l\theta'' + 2l'\theta' + gsin(\theta) = 0. \quad (3)$$

Notice that if we have the pendulum's length constant, then l' becomes zero, and we have the equation of a simple pendulum. In addition, we can see that l' plays the role of a friction factor (see Blanchard, Devaney and Hall (1998)) when it is positive, i.e. when l' grows. Obviously the length of the pendulum's cord cannot increase arbitrarily in our problem.

A heuristic strategy to decrease the pendulum's oscillations would be to lengthen quickly the pendulum cord ($l' > 0$) when the pendulum reaches its maximum speed (θ' maximum); thus we get the desired friction term for the damping; and to shorten the cord ($l' < 0$) when the speed is zero ($\theta' = 0$), thereby eliminating the negative term. In section 4 we will show numerically and graphically the effectiveness of this strategy.

2.2 Euler Lagrange equations

We now show some theory and results related to the pendulum problem. We use the following theorems to find the Euler equation. The proofs of the next theorems are shown in Gelfand and Fomin (1991) and in Hsu and Meyer (1968).

Theorem 2.1. *If $f(x)$ and $l(x)$ are continuous functions in $[a, b]$, and if*

$$\int_a^b [f(x)h(x) + l(x)h'(x)]dx = 0$$

for all differentiable function $h(x)$ in (a, b), such that $h(a) = h(b) = 0$, then $l(x)$ is differentiable, and $l'(x) = f(x)$ for all $x \in [a, b]$.

Theorem 2.2. *A necessary condition for a differentiable functional $J(y)$ to have an extremum at $y = \widehat{y}$ is that its variation is zero for $y = \widehat{y}$. It is $\delta J(h) = 0$ for $y = \widehat{y}$ and for all admissible h.*

A simple variational problem can be formulated as follows. Let $F(x, y, z)$ be a function with first and second continuous partial derivatives. For all continuous differentiable functions $y(x)$, $a \leq x \leq b$ that satisfy $y(a) = A, y(b) = B$; the problem is to find the extremum of the functional $J(y) = \int_a^b F(x, y, y')dx$.
If we increment $y(x)$ by $h(x)$, a condition for which the function $y(x)+h(x)$ satisfies the boundary conditions is $h(a) = h(b) = 0$. The increment in J, consequent to that in $y(x)$, is:

$$\Delta J = J(y + h) - J(y) = \int_a^b [F(x, y + h, y' + h') - F(x, y, y')]dx$$

Using Taylor's theorem we have

$$\Delta J = \int_a^b [F_y(x, y, y')h + F_{y'}(x, y, y')h']dx + ...$$

where F_y and $F_{y'}$ are the partial derivatives, and the ellipsis denotes the terms of higher order than 1 with respect to h and h'. The integral in the last equation is the linear part of ΔJ, called the variation of $J(y)$. By Theorem 2.2, a necessary condition for $y = y(x)$ to be an extremum point of J is that:

$$\delta J = \int [F_y h + F_{y'} h'] dx = 0$$

for all admissible functions h. Theorem 2.1 implies that

$$F_y - \frac{d}{dx} F_{y'} = 0$$

This last result is called Euler equation. Then we have proved that if $J(y)$ is a functional of the form

$$J(y) = \int_a^b F(x, y, y') dx$$

defined in a set of functions $y(x)$ with continuous first derivatives in $[a, b]$ and satisfying the boundary conditions $y(a) = A$ and $y(b) = B$, then, a necessary condition for $J(y)$ to have an extremum at $y(x)$ is that $y(x)$ satisfies the Euler equation. The solutions of the Euler equation are called extremals.

We now consider a problem of a different type, a variational problem with subsidiary conditions, which can be stated as follows. Find the functions $y(x)$, $z(x)$ for which the functional

$$J[y, z] = \int_a^b F(x, y, z, y', z') dx$$

has an extremum, where the admissible functions satisfy boundary conditions and the constraint

$$c(x, y, z, y', z') = 0$$

Then we have the following theorem.

Theorem 2.3. *Given a functional*

$$J[y, z] = \int_a^b F(x, y, z, y', z') dx,$$

where $y = t(x)$ and $z = z(x)$ must satisfy the constraint

$$c(x, y, z, y', z') = 0$$

and the boundary conditions

$$y(a) = A_1, y(b) = B_1, z(a) = A_2, z(b) = B_2$$

Let $(y^*(x), z^*(x))$ be an extremal of J. If c_y and c_z don't cancel simultaneously in some point of the manifold defined by $c(x) = 0$, then there exists a function $\lambda(x)$, such that $(y^*(x), z^*(x))$ is an extremal of the functional

$$\int_a^b [F + \lambda(x) c] dx,$$

i.e. it satisfies the Euler equations

$$F_y + \lambda c_y - \frac{d}{dx}F_{y'} = 0; \quad F_z + \lambda c_z - \frac{d}{dx}F_{z'} = 0.$$

This theorem can be generalized to n functions f_i $(i = 1, 2, ..., n)$, k restrictions c_i $(i = 1, 2, ..., k)$ and higher derivatives like it is shown in the next theorem which will be used to deduce the equations that control the pendulum system.

This theorem can be generalized to n functions f_i $(i = 1, 2, ..., n)$, k restrictions g_i $(i = 1, 2, ..., k)$ and higher derivatives like it is shown in the next theorem which will be used to deduce the equations that control the pendulum system.

Theorem 2.4. *Given a functional*

$$J[y, z] = \int_a^b F(x, y, z, y', z', y'', z'')dx,$$

where $y = t(x)$ and $z = z(x)$ must satisfy the constraint

$$c(x, y, z, y', z', y'', z'') = 0$$

and the boundary conditions

$$y(a) = A_1, y(b) = B_1, z(a) = A_2, z(b) = B_2$$

Let J have an extremum for the curve $(y^*(x), z^*(x))$. If c_y and c_z don't cancel simultaneously in some point of the manifold defined by $c(x) = 0$, then there exists $\lambda(x)$, such that $(y^*(x), z^*(x))$ is an extremal of

$$\int_a^b [F + \lambda(x)c]dx,$$

i.e. it satisfies the equations

$$\phi_y - \frac{d}{dx}\phi_{y'} + \frac{d^2}{dx^2}\phi_{y''} = 0, \quad \phi_z - \frac{d}{dx}\phi_{z'} + \frac{d^2}{dx^2}\phi_{z''} = 0$$

2.3 Functional energy for the pendulum in the plane

Equation (3) is equivalent to the following one:

$$\frac{1}{2}[(l(s)\theta'(s))^2 + gl(s)(1 - cos(\theta(s)))]|_0^T = gl(s)|_0^T - \int_0^T [gcos(\theta(s))l'(s) + (\theta'(s))^2 l(s)l'(s)]ds \qquad (4)$$

In the left-hand side of (4) we find part of the sum of the kinetic energy (not including the radial component) and the potential energy of the system. We call

this the oscillation energy. We can verify this equation by differentiating (4) with respect to T and simplifying to get (3).
From (4) we can see that the term $gl(\theta)|_0^T$ becomes null in the case $l(0) = l(T)$, i.e. when the pendulum has the same length at the beginning and at the end of the movement. i.e. in the time $t = 0$, and $t = T$. Considering these facts, we have:

$$\frac{1}{2}[(l(s)\theta'(s))^2 + gl(s)(1 - cos(\theta(s)))]|_0^T = \\ -\int_0^T [gcos(\theta(s))l'(s) + (\theta'(s))^2 l(s)l'(s)]ds \qquad (5)$$

This equation represents the oscillation energy of the pendulum. In order to reduce the oscillation energy of the pendulum, we need to minimize the value of the integral of this equation.
Using theorem 2.4 we have:

$$\begin{aligned} F &= F(t, \theta, l, \theta', l', \theta'', l'') \\ &= -gcos(\theta(s))l'(s) - (\theta'(s))^2 l(s)l'(s) \end{aligned}$$

and

$$\begin{aligned} G &= G(t, \theta, l, \theta', l', \theta'', l'') \\ &= l(t)\theta''(t) + 2l'(t)\theta'(t) + gsin(\theta(t)) \end{aligned}$$

The Euler equation then becomes:

$$F_l + \lambda G_l - \frac{d}{dt}(F_l' + \lambda G_{l'}) + \frac{d^2}{dt^2}(F_{l''} + \lambda G_{l''}) = 0 \qquad (6)$$

$$F_\theta + \lambda G_\theta - \frac{d}{dt}(F_{\theta'} + \lambda G_{\theta'}) + \frac{d^2}{dt^2}(F_{\theta''} + \lambda G_{\theta''}) = 0 \qquad (7)$$

Taking the derivatives in equations 6 and 7 we have:

$$2l\theta'\theta'' - \lambda\theta'' - gsin(\theta)\theta' - 2\lambda'\theta' = 0 \qquad (8)$$

$$2ll'\theta'' + gl'sin(\theta)\theta' + \lambda gcos(\theta)\theta' + 2(l')^2\theta' + 2ll''\theta' + \lambda l'' + l\lambda'' = 0 \qquad (9)$$

Therefore the equations for $\theta(t)$ and $l(t)$ that minimize the functional

$$J(\theta, l) = -\int_0^T [gcos(\theta(s))l'(s) + (\theta'(s))^2 l(s)l'(s)]ds \qquad (10)$$

restricted to

$$l(t)\theta''(t) + 2l'(t)\theta'(t) + gsin(\theta(t)) = 0 \tag{11}$$

are given by equations (8) and (9). The function $\theta(t)$ satisfies the conditions $\theta(0) = a, \theta'(0) = b, \alpha \leq l(t) \leq \beta$. These conditions must be considered when we solve the system of equations. We have a non-linear second order system of equations with unknowns (θ, l, λ), and we don't have initial conditions for λ. Another observation is that Euler's equations require extremal functions to have continuous derivatives, that is, for this system θ and l must be differentiable, and this may not be our case.

3 Pontryagin's principle and optimum control

The Pontryagin's principle is a result for optimal control problems where the control can be a discontinuous bounded function. This principle allows us to find extremals for a functional of the form:

$$J = \int_0^T F_0(x, u, t)dt \tag{12}$$

subject to

$$x_i' = F_i(x, u, t) \qquad i = 1, 2, ..., n \tag{13}$$

where

$$x = (x_1(t), x_2(t), ..., x_n(t))$$

$$u = (u_1(t), u_2(t), ..., u_m(t))$$

and t is a time variable. Using theorem 4 and Lagrange multipliers λ_i we build the new functional

$$J^* = \int_0^T F_0 + \sum_{i=1}^{n} \lambda_i(F_i - x_i')dt.$$

An extremal for (12) subject to (13) is an extremal for J^*, i.e. the functions $x(t)$ and $u(t)$ that satisfy the Euler equations for J^*. We define

$$H = F_0 + \sum_{i=1}^{n} \lambda_i F_i \qquad \text{(Hamiltonian)} \tag{14}$$

Let's assume that F_0 and F_i do not depend explicitly on t, i.e. $H = H(x, u, \lambda)$. Now we have:

$$J^* = \int_0^T H - \sum_{i=1}^{n} \lambda_i x_i' dt. \tag{15}$$

Euler equations of (15) for variables x_i are:

$$\frac{\partial H}{\partial x_i} - \frac{d}{dt}(-\lambda_i) = 0,$$

solving we have:

$$\lambda_i' = -\frac{\partial H}{\partial x_i} \qquad i = 1, 2, ..., n \tag{16}$$

We don't consider the control u_j in equation (16) because u_j could be discontinuous. Instead we consider the effect of the functional when small changes in the control occur, i.e. if we consider

$$\Delta J^* = J^*(x, u + \delta u, \lambda) - J^*(x, u, \lambda)$$

we have

$$\Delta J^* = \int_0^T \sum_{j=1}^m [H(x; u_1, u_2, ..., u_j + \delta u_j, ..., u_m; \lambda) - H(x, u, \lambda)]dt \tag{17}$$

In order for u to become the optimum, we need that $\Delta J^* \geq 0$ for each admissible control $u + \delta u$.
Then we have:

$$H(x; u_1, u_2, ..., u_j + \delta u_j, ..., u_m; \lambda) \geq H(x, u, \lambda) \tag{18}$$

for each admissible δu_j and $j = 1, 2, ..., m$, i.e. for an optimum control, H is minimized with respect to control variables $u_1, u_2, ..., u_m$. Now we can enunciate the Pontryagin's principle.

Theorem 3.1. *(Pontryagin's principle.) Given a functional*

$$J = \int_{t_1}^{t_2} F_0(x, u, t)dt$$

where F_0 is continuous function by parts in u,

$$x = (x_1(t), x_2(t), ..., x_n(t))$$

$$u = (u_1(t), u_2(t), ..., u_n(t))$$

Given a control function u^*, $t_1 \leq t \leq t_2$ with its corresponding trajectory $x^*(t)$ subject to

$$x_i' = F_i(x, u, t) \quad (i = 1, 2, ..., n)$$

where F_i is continuous by parts with respect to u. A necessary condition to minimize J is that a non-null continuous function $\lambda(t)$ must exist such that for each time t, the Hamiltonian is minimized in the optimum control with respect to another control. i.e.

$$H(x, u^*, t, \lambda) \leq H(x, u, t, \lambda)$$

Applying the previous theory to our problem, we have the following functional

$$J(\theta, l) = -\int_0^T [gcos(\theta(s))l'(s) + (\theta'(s))^2 l(s)l'(s)]ds \tag{19}$$

subject to

$$l(\theta)\theta''(t) + 2l'(t)\theta'(t) + gsin(\theta(t)) = 0 \tag{20}$$

$$\alpha \leq l \leq \beta \tag{21}$$

From equation (20) we have

$$l' = \frac{-l\theta'' - gsin(\theta)}{2\theta'}$$

Replacing in functional (19) we have

$$J = -\int_0^T [gcos(\theta)(\frac{-l\theta'' - gsin(\theta)}{2\theta'}) + (\theta')^2 l(\frac{-l\theta'' - gsin(\theta)}{2\theta'})]ds$$

$$J = \int_0^T [\frac{gcos(\theta)\theta'' l}{2\theta'} + \frac{gsin(\theta)\theta' l}{2} + \frac{\theta'\theta'' l^2}{2} + \frac{g^2 cos(\theta)sin(\theta)}{2\theta'}]ds \tag{22}$$

Introducing a change of variable:

$$\theta = \theta_1, \theta_1' = \theta_2, \theta_2' = \theta_3 \tag{23}$$

From equations (22) and (23) we have:

$$J = \int_0^T [\frac{gcos(\theta_1)\theta_3 l}{2\theta_2} + \frac{gsin(\theta_1)\theta_2 l}{2} + \frac{\theta_2\theta_3 l^2}{2} + \frac{g^2 cos(\theta_1)sin(\theta_1)}{2\theta_2}]ds$$

subject to

$$\theta_1' = \theta_2, \;\; \theta_2' = \theta_3$$

Now we have the functional

$$J^* = \int_0^T [\frac{gcos(\theta_1)\theta_3 l}{2\theta_2} + \frac{gsin(\theta_1)\theta_2 l}{2} + \frac{\theta_2\theta_3 l^2}{2}dt + \frac{g^2 cos(\theta_1)sin(\theta_1)}{2\theta_2} + \lambda_1(\theta_2 - \theta_1') + \lambda_2(\theta_3 - \theta_2')]dt$$

from (14) the Hamiltonian is

$$H = \frac{gcos(\theta_1)\theta_3 l}{2\theta_2} + \frac{gsin(\theta_1)\theta_2 l}{2} + \frac{\theta_2\theta_3 l^2}{2} + \frac{g^2 cos(\theta_1)sin(\theta_1)}{2\theta_2} + \lambda_1\theta_2 + \lambda_2\theta_3 \tag{24}$$

Solutions for $\theta_1, \theta_2, \theta_3, l, \lambda_1, \lambda_2$ are given by solutions of equation (16) and minimizing (24) with respect to l.
Replacing equivalent values for $\theta_1, \theta_2, \theta_3$, (24) changes to

$$H = \frac{gcos(\theta)\theta'' l}{2\theta'} + \frac{gsin(\theta)\theta' l}{2} + \frac{\theta'\theta'' l^2}{2} + \frac{g^2 cos(\theta) sin(\theta)}{2\theta'} + \lambda_1\theta' + \lambda_2\theta'' \qquad (25)$$

From this last equation, we see that l appears only in the first three terms, then we need only to minimize with respect to l the following functional

$$H^* = \frac{gcos(\theta)\theta'' l}{2\theta'} + \frac{gsin(\theta)\theta' l}{2} + \frac{\theta'\theta'' l^2}{2}$$

Using a change of variables and rewriting this last equation we have

$$H^* = l(Al + B),$$

where

$$A = \frac{\theta'\theta''}{2}, \quad B = \frac{gcos(\theta)\theta''}{2\theta'} + \frac{gsin(\theta)\theta'}{2}$$

We can see that H^* is a quadratic function of l; since $\alpha \leq l \leq \beta$, $\alpha, \beta > 0$, we can minimize H^* according to the values of A y B.

First case

For $A \geq 0$ we have a parabola that opens upward. The values for l that minimize H^*, depend only on B. If $B \geq 0$, the intercepts are 0 and $-B/A$, then we have that $l = \alpha$ minimizes H^* (Figure 2). If $B < 0$, the intercepts are 0 and $-B/A$, then we have to consider the position $-B/2A$.

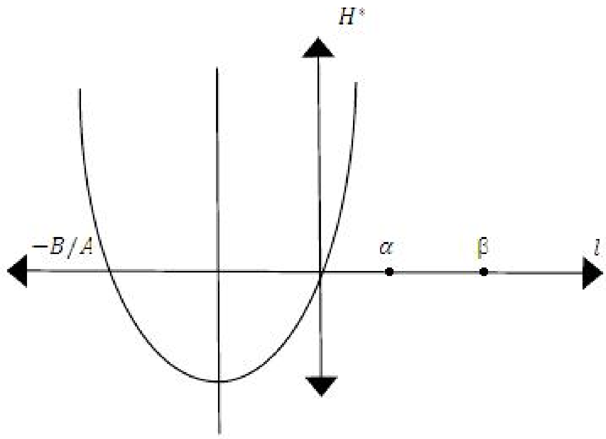

Figure 2 - l vs H^* for the case $A \geq 0, B \geq 0$.

Second case

For $A < 0$, we have a parabola that opens downward passing by the origin. Again, in order to minimize H^* we have to know the values taken by l which depend on B.
In summary we obtained a control strategy l which depends only on A and B, which depend only on $\theta, \theta', \theta''$, that is, we have a feedback control and not a control that depends on independent variable t.

4 Numerical analysis

We have two strategies to control the oscillations of a pendulum, by variations in the pendulum's length, the heuristic strategy and the optimal strategy founded using Pontryagin's principle.
In this section we will show graphically, using the program Matlab, the results of the heuristic strategy and the optimal strategy. We use the fourth order Runge Kutta method (see Suli and Mayers (2003)) to solve the equation that controls the pendulum in the plane using both strategies. We will graph the oscillation energy and the oscillation angle θ of the pendulum system and will see how this strategies diminish the energy of the system and the amplitude of the oscillations making the pendulum stabilize. We will make comparisons between the heuristic and the optimal strategies. we will also graph the optimal control $l(t)$. Finally we'll make comparisons between the heuristic control and the optimal control.

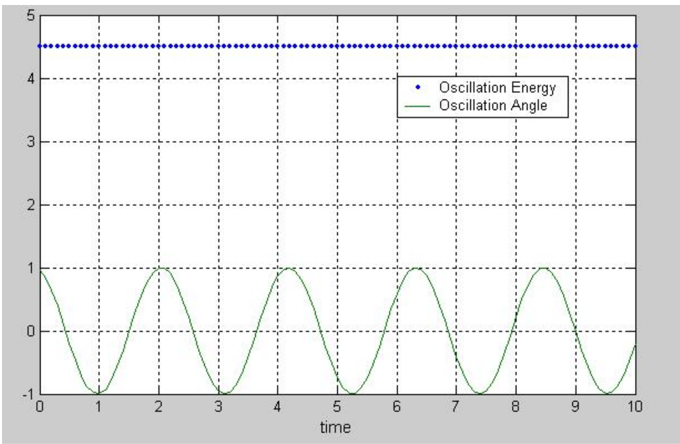

Figure 3 - Energy and Oscillation angle of a pendulum with length 1.

Figure 3 shows the energy of the pendulum and the angle θ as functions of time, where h is the step size of the numerical program. As initial conditions we have: $h = 0.01, \theta = 1, \theta' = 0$ and $l = 1$. In this case, we can see that the energy of the system is constant.

In Figure 4 and 5 we show the effect of the heuristic strategy (Figure 4) and the optimum strategy of Pontryagin (Figure 5) applied to the pendulum of the Figure 3, changing the length of the cord up to 20 percent ($1 \leq l \leq 1.2$). The results show that the energy and the amplitude of oscillations decrease in time (the program is implemented for a step size of $h = 0.1$).

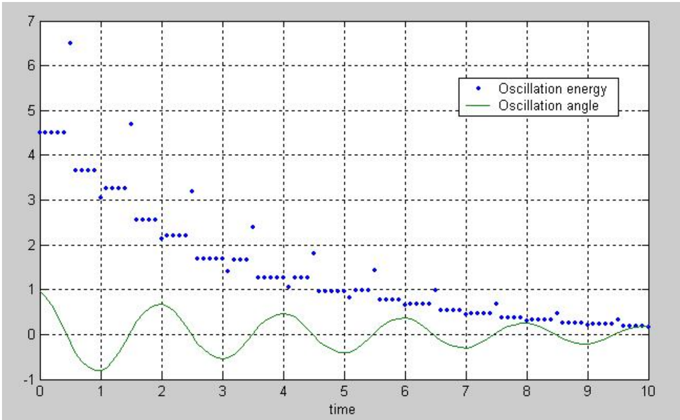

Figure 4 - Heuristic strategy for a step size of $h = 0.1$.

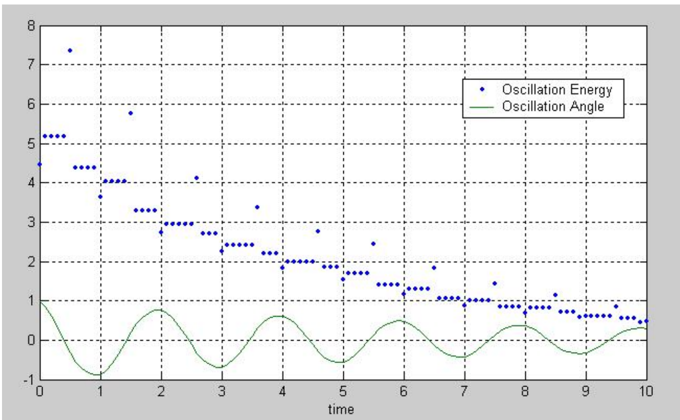

Figure 5 - Optimal strategy of Pontryagin for a step size of $h = 0.1$.

We can also see that the heuristic strategy seems to be more efficient than the optimal strategy, since the energy decreases faster for the first case.
If we shorten h more, it can be shown that the optimum strategy approaches to the heuristic strategy. In Figures 6 and 7, implemented for $h = 0.05$, we show a comparison of both strategies.

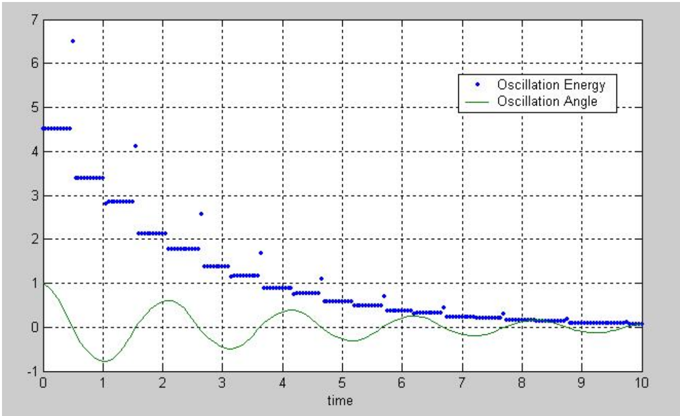

Figure 6 - Heuristic strategy for a step size of $h = 0.05$.

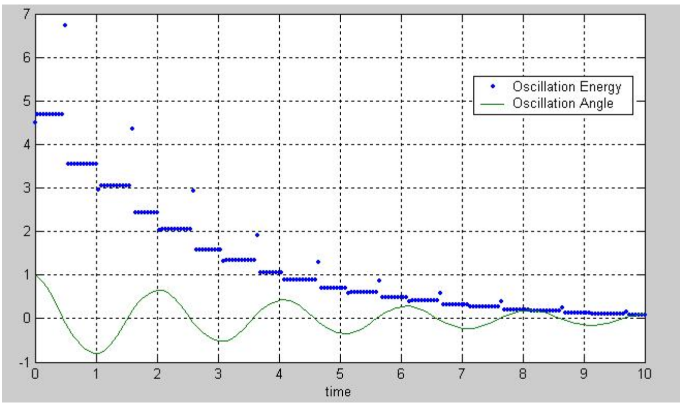

Figure 7 - Optimal strategy of Pontryagin for a step size of $h = 0.05$.

If we shorten h more and more, it can be shown that the optimal strategy converges to the heuristic strategy that is, the optimal control turned out to be the heuristic control. Figures 8 and 9, implemented for $h = 0.01$, show that both strategies are practically the same.

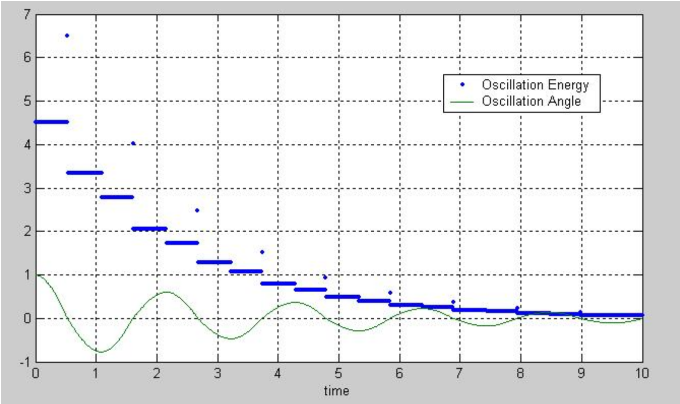

Figure 8 - Heuristic strategy for a step size of $h = 0.01$.

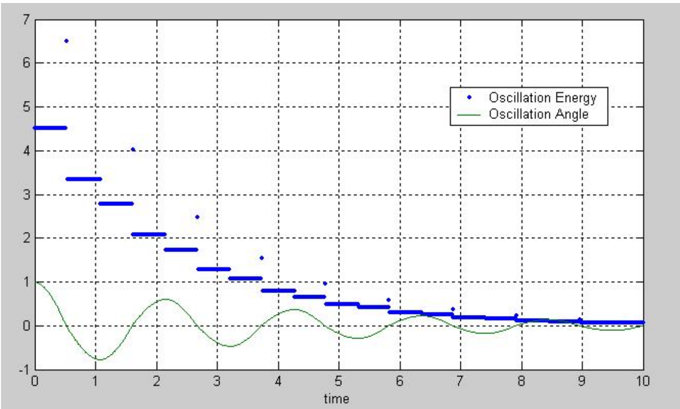

Figure 9 - Optimal strategy of Pontryagin for a step size of $h = 0.01$.

The next three Figures (10,11,12) represent the angular oscillations of both strategies in the same graph. They show how the optimal strategy of Pontryagin converges to the heuristic strategy as the step size becomes shorter, the strategies are the same. As in the previous figures, the greater amplitude corresponds to the heuristic strategy.

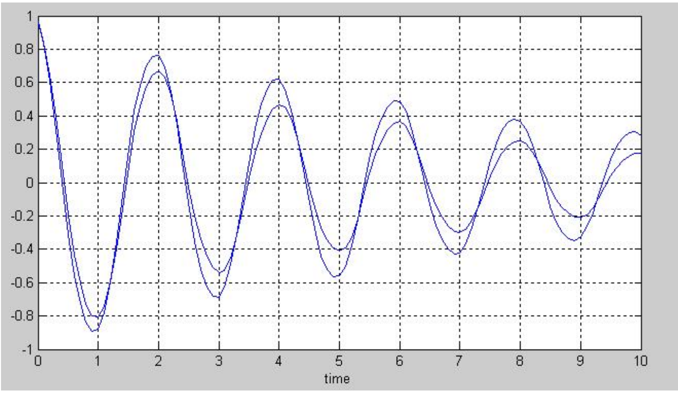

Figure 10 - Angular oscillations of both strategies for a step size of $h = 0.1$.

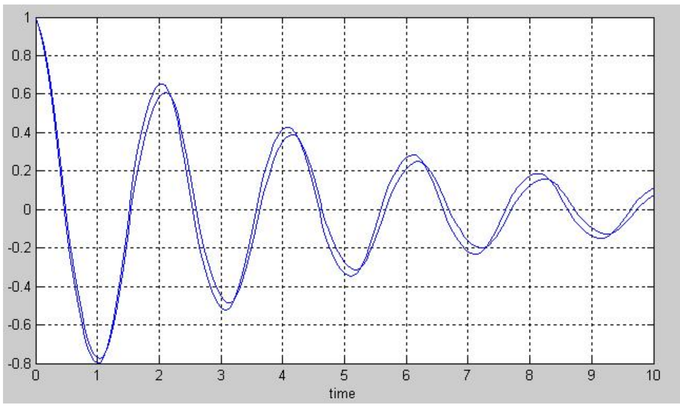

Figure 11 - Angular oscillations of both strategies for a step size of $h = 0.05$.

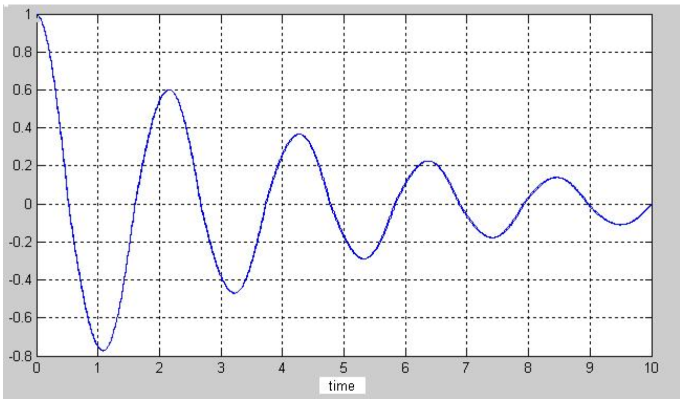

Figure 12 - Angular oscillations of both strategies for a step size of $h = 0.01$.

Finally Figure 13 shows the optimal control. As we can see the control function l takes the only values 1 and 1.2. The control l change the values of $l = 1$ to $l = 1.2$ when θ' is maximum $(\theta = 0)$ and l change from $l = 1.2$ to $l = 1$ when θ' is zero.

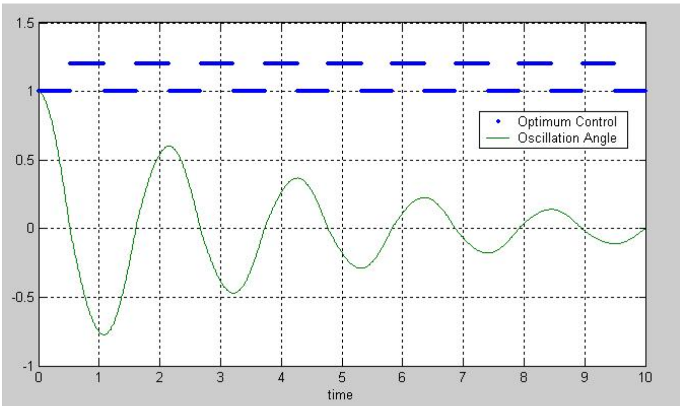

Figure 13 - Optimum strategy of Pontryagin.

5 Conclusions and future work

The issues discussed in this paper have led us to the following conclusions:

Using Pontryagin's principle, we found a optimal strategy to control the pendulum's planar oscillation, by variations in the pendulum's length. The numerical computations show that this optimal strategy turns out to be a known heuristic strategy in which the control enlarge its length when the mass of the pendulum reaches its maximum speed and reduce its length when it reaches its minimum speed.

The optimal strategy of Pontryagin we found is given by a feedback control, that is, it is not a control which depends directly of the time. The control with feedback is more robust, because it can be adjusted to deviations or differences between reality and mathematical model. This control receives updates continuously from the real state of the system.

The optimum control is given by a bang bang control (see Jacobs (1974)), that is, it takes only its lower and upper bounds. The control function l only takes the values α, β, as the figures show (in our examples $\alpha = 1$, $\beta = 1.2$).

Using the optimal strategy of Pontryagin, it is shown numerically that we can decrease the energy of the pendulum as much as we want, i.e. we can stabilize the pendulum.

On the other hand, the issues discussed in this paper have led us the following proposals for future work:

1) Find optimum strategies to control the oscillations of a pendulum which is moving in three dimensions.

2) Implement simulations to illustrate the effect of the optimal strategy of Pontryagin for the pendulum in three dimensions.

3) Apply Pontryagin's Principle for more complicated systems, like the double pendulum.

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DELGADO, M.; PORTNOY, A.; FÚQUENE P.; J. A. Controlando as oscilações de um pêndulo de comprimento variável. *Rev. Bras. Biom.,* São Paulo, v.24, n.4, p.66-84, 2010.

- RESUMO: Um método eficaz para parar um pêndulo oscilações planar por variações no comprimento do pêndulo é encontrado. Isto é conseguido a través da carcaça do problema como um problema de controle ótimo. Equações que governam o pêndulo são deduzidas e usando estas equações a energia de oscilação do pêndulo é encontrado. O problema se torna um problema variacional com restrições, em que um funcional que representa a energia de oscilação do pêndulo deve ser minimizado. Usando o princípio Pontryagin, as soluções ótimas são encontradas. Finalmente, a eficácia das estratégias encontradas é ilustrada graficamente, comparações analíticas e numéricas são feitas.

- PALAVRAS-CHAVE: PALAVRAS-CHAVE: Pêndulo de comprimento variável; a energia de oscilação; Pontryagin princípio.

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