AN ELEMENTARY DEMONSTRATION OF THE FRIENDSHIP THEOREM USING GRAPHS WITH AN APPLICATION IN EXPERIMENTAL DESIGNS

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▪ ABSTRACT: One proof of the friendship theorem, a classical result in combinatorics, is presented. Graphs are intensively used to explain all the steps of the demonstration and thus make it more intuitive. An application in experimental designs is presented.

▪ KEYWORDS: Regular graphs; eigenvectors; partially balanced designs.

1 Introduction

The Friendship Theorem is a classical result that is considered by many people one of the expressions of Beauty in Mathematics. A statement that everyone understands, with several different proofs, using graph theory, linear algebra, and combinatorics. Many of these demonstrations have various ideas that can be used to deal which other different problems in mathematics. The Friendship Theorem was so popular that it admits the following formulation in terms of an everyday context that can be understood by everyone.

Friendship Theorem: In a party with n people, where any two of them have exactly one friend in common, there is one, and only one person, who is a friend with everyone else in the party.

This theorem can and was often presented, as a theorem on graph theory. People in the party are considered as vertices and if one person is a friend with another person then the two corresponding vertices will be connected by an edge (in this case the vertices are said to be adjacent). We will refer to people as vertices and vertices as people interchangeably and the adjacency relationship as a friendship between two of them. In terms of graph theory, the theorem can be stated as:

Friendship Theorem: If in a graph G with a finite number of vertices we have the property that given any two vertices there is only one vertex that is adjacent to both, then there is a single vertex that is adjacent to all vertices of the graph.

This theorem was the subject of an excellent article in Portuguese (Casarin and Tomei, 1987) "Uma Demostração Elementar do Teorema da Amizade". The proof presented in this article is elementary, however, its reading is by no means simple and direct. The authors of

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this article suggest the readers must follows the steps of demonstration making drawings and this is not so simple for the reader. The approach essentially uses the language of set theory, which in our view makes reading less direct and less accessible to a more general audience.

The novelty of our new paper is to remake the proof presented by Casarin and Tomei (1987) in a more intuitive way, using graphs and their respective graphical representations, as intensively as possible. We hoped that this approach will make the demonstration much more didactic and intuitive and can be understood by a wider audience, in particular, professionals that work with biometry. An application of the theorem to an experimental design problem is presented. The use of graphs in the theory of experimental designs is a recent area of research, which has obtained interesting results, in particular, in obtaining optimal designs. As a reference of this subject we can mention "Combinatorics of Optimal Designs" (Bailey and Cameron, 2009) and "Using Graphs to Find the Best Block Designs" (Bailey and Cameron, 2011).

Another good reference related to the Friendship Theorem in Portuguese is the work of Calegari (2018). It is strongly based on general graph theory, such as the use of cycles, which makes his work undoubtedly a reading aimed at a specialized audience in graph theory. It is a different proposal from ours, which is to present a demonstration to a wider audience, based on an intuitive and geometric approach. For more complex proofs see Walker (2017) e Chatterjee (2014).

2 Proof of the Friendship Theorem

Let's enumerate the vertex of graph G from 1 to n. The vertices of graph G are the people in the party and their friendship relationships, so we will consider the common friend of 2 and 3 to be vertex 1. Here we have two possibilities given by figures 1 and 2.

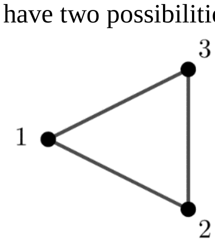

Figure 1 – Vertex 2 e 3 are friends.

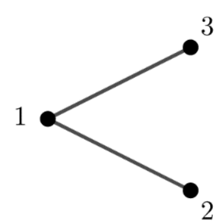

Figure 2 – Vertex 2 e 3 are not friends.

For the case of Figure 2, we will take the common friend of 1 and 2 and call it vertex 4, as shown in Figure 3.

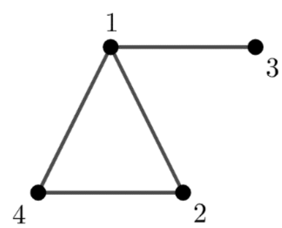

Figure 3 – Vertex 1, 2 e 4 are friends.

In the same way, taking the common friend of 1 and 3 and naming it vertex 5, we have figure 4.

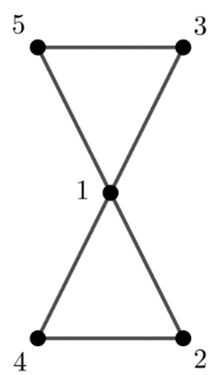

Figure 4 – Vertex 1, 3 e 5 are friends.

This construction can be extended until all friends of 1 are obtained. If $2,\ 3, \ldots, m-1,\ m$ are all friends of 1, m is necessarily odd, that is, $m = 2k + 1$. Let's take the subgraph of G formed by 1 and all of its friends and their respective friendship relations (edges). This subgraph can be represented by triangles centered at 1, which we will call a bouquet of triangles. Note that in this subgraph only the friendship relations of person 1 and the friendship relations between the friends of 1 are considered. If 1 is friend with 2 and 3, only, there is one triangle (figure 5).

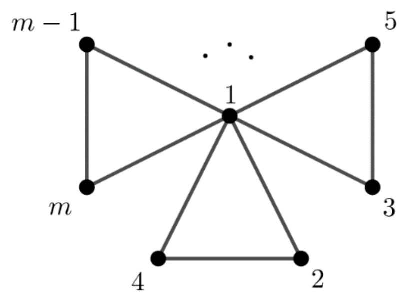

Figure 5 - All vertex are friends of 1.

Let's now do the same construction for each of the vertices $2,3,\ldots,m-1,m$. That is, for the i-th friend of 1 we will take all the friends of i. Let's call $A_i = 2,3,..,m$ the set of friends of the i-th friend of 1, excluding 1 and the common friend of 1 and i. We will also call A_i the set of friends of 1.

The sets A_i are disjoint. If p is in A_i and A_j with i and j different from 1, then p is friend of i and friend of j, but the only common friend of i and j is 1. A_1 and A_i are also disjoint because otherwise, $p \in A_i$ would be friend of 1 and friend of i but different from common friend of 1 and i in A_1. We also have that no person in A_i for i greater than one is friend with 1.

We now have a subgraph of G formed by several bouquets of triangles centered on these vertices as shown in figure 6.

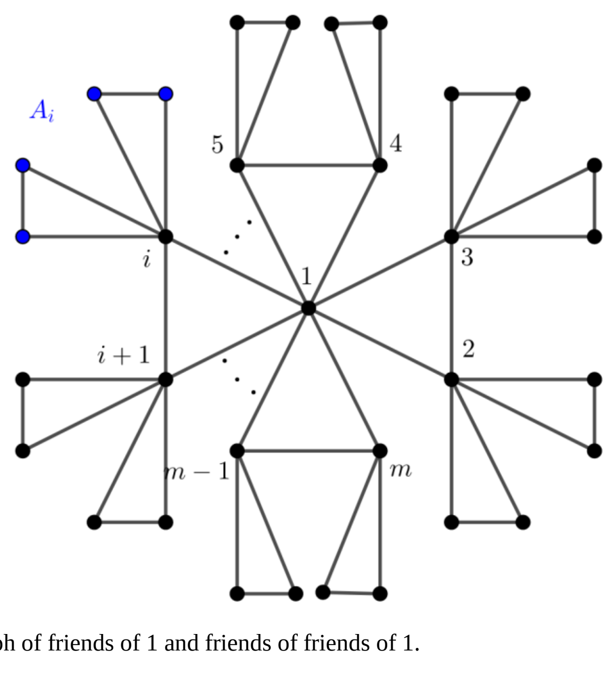

Figure 6 - Subgraph of friends of 1 and friends of friends of 1.

If this process can be reproduced indefinitely, this give us the idea that the Theorem is not true when one assumes an infinite number of people at the party, that is, for the case where the graph G is infinite. (Such graph can be seen at: https://www.theoremoftheday.org/CombinatorialTheory/Friendship/TotDFriendship.pdf).

Affirmation 2.1. *The subgraph in Figure 6 represents all the people at the party*.

Demonstration of affirmation 2.1. Let v be a person who is not represented in the graph of figure 6. Take the common friend of v and 1. Note that v is not a friend of 1 because all friends of 1 are represented in the subgraph. Since the common friend of 1 and v is a friend of 1, say the j-th friend of 1, so v is a friend of j and therefore is in the bouquet centered on j, which is a contradiction to v not being represented in the subgraph.

Then we have the subgraph (figure 6) that contains all vertices of graph G but not necessarily all edges. For the proof of the theorem we have to show that the sets of friends of friends of 1 are empty, that is, A_i is empty for $i = 2, 3, \dots, m$. In this case, the graph G is equal to the bouquet of triangles centered on 1 and therefore 1 is friend of everyone at the party.

If there is one person who is friend with exactly two people then the theorem is true. Let's name this person by 1 and 2 and 3 his friends. In this case, we only have the sets A_2 and A_3, as in figure 7.

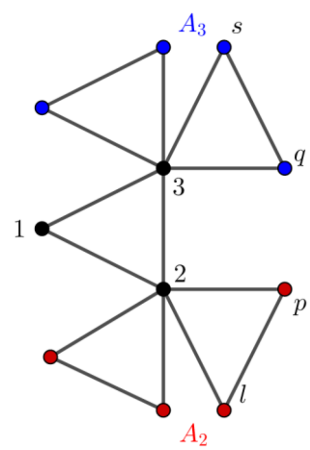

Figure 7 – Sets A_2 and A_3.

Let's show that one of them is empty. If $p \in A_2$ and $q \in A_3$ then their mutual friend cannot be 1 because in this case 1 and 3 would have 2 and q as mutual friends, and also 1 and 2 would have 3 and p as mutual friends. It cannot be 2 because in this case, 3 and q would have two friends in common, 3 and s. Likewise, it cannot be 3 because in this case, 2 and p would have two friends in common, 3 and l. Therefore, one of the sets A_2 or A_3 is empty. Assuming A_3 empty, 2 is friend with everyone at the party.

Therefore, to continue the proof of the theorem we can assume that every person at the party is friend with at least 4 people.

Other interesting properties related to the subgraph of figure 5 are:

(a) *If i and j are two friends of 1 that are not friends with each other, then A_i and A_j have the same number of vertices.*

Demonstration of item (a). Assuming A_i and A_j are not empty. The only common friend, named p, between i and q in A_j is some person of A_i because p cannot be 1 (no one of A_j is a friend of 1); nor can it be r (since 1 and r already have i as a mutual friend). As p must be friend with i, there is only one possibility, that p is in A_i, as in figure 8.

Taking another person l in A_j other than q the mutual friend of i and l in A_i, say person s, will be different from p. In fact, if $l = p$ then p and j have q and l as mutual friends as in figure 9.

It follows that the number of people in A_j is less than the number of people in A_i. Making the same argument using symmetry, follows the equality of the number of people in A_i and in A_j. If A_j is empty we will show that A_i is also empty. Taking p in A_i the mutual friend of j and p could only be 1 or r, the mutual friend of 1 and j. But p is not friend with 1, so this friend would be r, but in this case, 1 and p would have i and r as common friends.

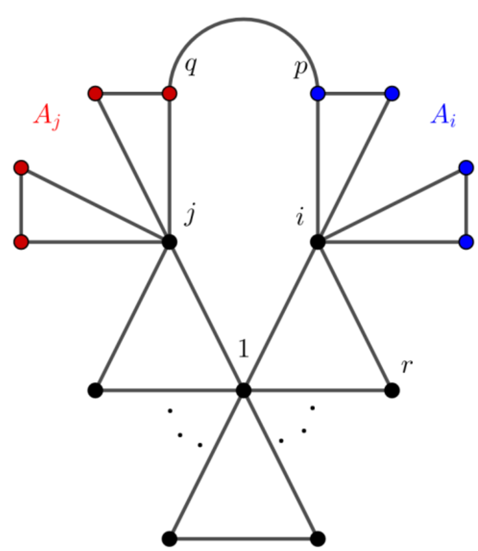

Figure 8 - Relationships between i friends and j friends.

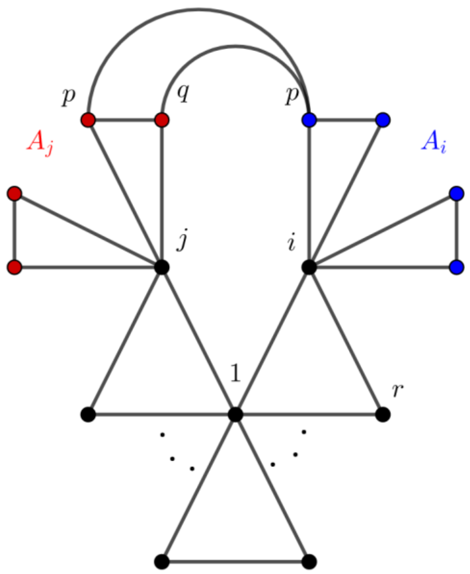

Figure 9 - Relationships between i friends and j friends.

(b) *All sets* A_i, $i = 2, 3, \ldots, 2k + 1$, *have the same number of elements.*

Demonstration of item (b). Since we can assume that 1 has at least 4 friends, let i and j be friends of 1 who are not friends. So A_i and A_j have the same number of elements. Taking h as the common friend of 1 and j, h does not know i and therefore A_h has the same number of elements as A_i. Repeating the argument gives the result.

As the choice of person 1 was completely arbitrary, it can be said that for a person with more than two friends, all his friends are friends with the same number of people. It can then be concluded that all the people at the party have the same number of friends. Taking arbitrarily two people p and q they have a friend r in common. If r knows only two people, p and q, the theorem is true as we have seen. Remains the case where r knows more than two friends. In this case, all friends of r knows the same number of people and therefore p and q knows the same number of people. As they were taken as any participant of the party, everyone at the party knows the same number of people.

In terms of graph theory, we now have a much stronger information about graph G. All of its vertices have the same degree, that is, they are adjacent to the same number of vertices. A graph with this property is called a regular graph.

As we saw that the sets A_i had an even number of people, the degree of the vertices of G is an even number, say $2k$. From this information, it is possible to obtain the number of party participants. We have for the person 1, its $2k$ friends. For each friend i there are $2k$ friends, but friend 1 and the mutual friend of 1 and i have already been counted and, therefore, $2k-2$ friends remain. Therefore, the number of people at the party is $n = 1 + 2k + 2k(2k-2)$. Let's illustrate this situation by considering the subgraph with $k = 2$, that is, 1 is friend with two people and his friends also have two friends. We will then have a regular graph of degree 4 with $1 + 2.2 + 2.2(2.2 - 2) = 13$ vertices, see figure 10. Note that as can see in the graph that the common friend between 3 and 10 is 9, between 5 and 7, 13, and so on.

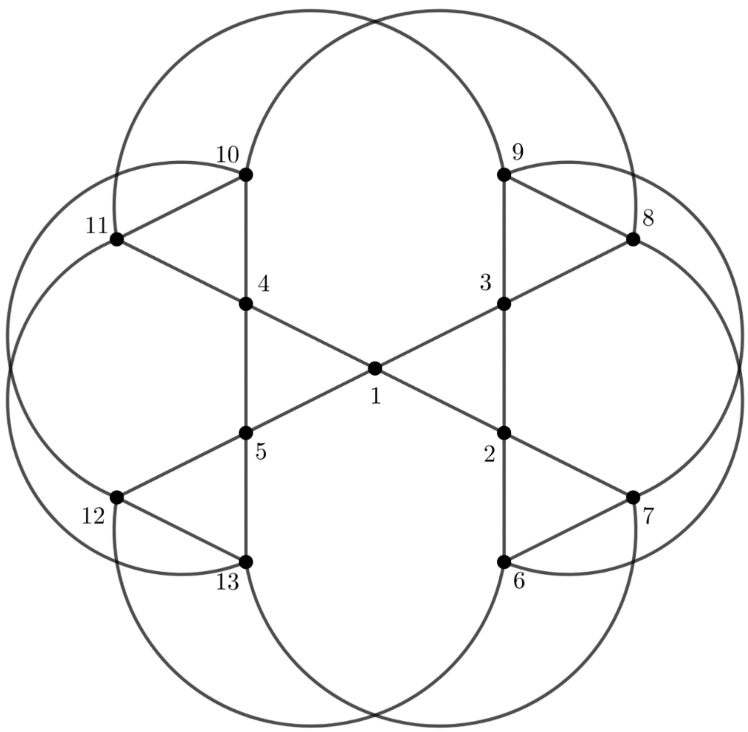

Figure 10 - Bijections between A_2, A_3, A_4 e A_5.

We have a problem here, the graph obtained does not satisfy the property because 11 and 13 don't have one common friend (figure 10).

We have the following theorem.

Theorem 2.1. *A regular graph of degree* $2k$ *with the property that two vertices have exactly one vertex adjacent to both exists only for the case* $k = 1$.

The proof of Theorem 2.1 will be based on linear algebra. Here is an interesting observation. The use of linear algebra to prove results in combinatorics is one of the most powerful methods used, but one question always remains, would such proof be possible using only counting arguments? An example is the following important result: for a balanced incomplete-block design, the number of blocks is greater than or equal to the number of treatments. This is the famous Fisher's Theorem whose proof is almost immediate using linear algebra, but there is still no proof using only combinatorics. See (Van Lint *et al.*, 2011, p.194), (Bailey and Cameron, 2011) and (Chatterjee, 2014).

To have a self-sufficient article, we will explain the demonstration presented in the reference article (Casarin and Tomei, 1987). For linear algebra see (Lima, 2014).

Proof of Theorem 2.1. Enumerating the people from 1 to n, the adjacency matrix M of the graph G is a square matrix of dimension equal to the number of vertices, defined by $M_{ij} = 1$ if i and j are adjacent and $M_{ij} = 0$ otherwise. Note that M is symmetric, with zeros on the diagonal and for the line i there is 1 for adjacent vertices and zero otherwise. Therefore, the sum of the elements of a row is $2k$. The matrix M^2 is such that if i is different from j we have $M_{ij}^2 = \sum_S M_{is} M_{js}$ which is the number of common friends of i and j and, therefore, equal to 1 and $M_{ii}^2 = 2k$ which is the number of people i knows, that is, the degree of vertex i. If J is a matrix with all entries equal to 1, we have the equality $M^2 = J + (2k - 1)I$ where I is the identity matrix. The vector $u = (1, 1, \ldots, 1)$is an eigenvector of the eigenvalue $n + 2k - 1$, and its multiplicity is 1. If $v = (v_1, v_2, \ldots, v_n)$ is such that $\sum_i v_i = 0$, then v is an eigenvector relative to the eigenvalue $2k - 1$ with multiplicity $n - 1$. The eigenvalues of M^2 are the squares of the eigenvalues of M with the same multiplicities, so the eigenvalues of M are, unless sign, $\sqrt{n + 2k - 1}$ with multiplicity 1 and $\sqrt{2k - 1}$, with multiplicity $n - 1$. Since the diagonal of M is formed of zeros and the sum of the eigenvalues is equal to the trace of M, which is zero, then:

$$\pm(n-1)\sqrt{2k-1} = \pm\sqrt{n-2k-1}$$
$$(\pm(n-1)\sqrt{2k-1})^2 = (\pm\sqrt{n-2k-1})^2$$
$$(n-1)^2(2k-1) = n - 2k - 1.$$

But we have to,

$$\mathrm{n = 1 + 2k + 2k(2k - 2) = 1 - 2k + 4k^2},$$

and therefore,

$$(2k + 4k^2)^2(2k - 1) = 4k^2$$
$$(4k^2 - 2k)^2(2k - 1) = 4k^2$$
$$(2k - 1)^2 = 1$$
$$k = 1.$$

Thus, proving Theorem 2.1.

As a consequence of Theorem 2.1, the only party where all participants have the same number of friends is a party with only 3 persons that are friends among them.

There remains, then, the possibility of one person being friend with exactly two people. Let's say person 2 is friends with 1 and 3. As we saw earlier, in this case, it can be considered that 3 has no other friends besides 1 and 2 and that 1 is the person at the party who is friends with everyone, thus proving the Theorem of Friendship. Graph G of the friendly relations between the people at the party is as in figure 11.

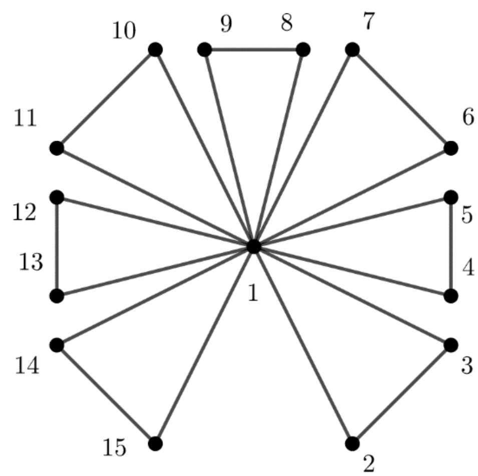

Figure 11 - Graph of the Friendship Theorem.

3 Graph theory and experimental designs.

It is possible to express a block design in terms of graphs. Define a graph with vertices given by treatments. Vertices $\boldsymbol{i}$ and $\boldsymbol{j}$, corresponding to treatments $\boldsymbol{i}$ and $\boldsymbol{j}$, will be connected by $\boldsymbol{\lambda_{ij}}$ edges if treatments $\boldsymbol{i}$ and $\boldsymbol{j}$ occur together in $\boldsymbol{\lambda_{ij}}$ blocks. This graph is called the design concurrency graph (Bailey and Cameron, 2011).

Consider a design with the following property: given two treatments $\boldsymbol{i}$ and $\boldsymbol{j}$ then either there is one and only one block that contains these two treatments or these two treatments do not occur together in any block. This design is binary and unbalanced. Note that the concurrency graph of this design has the property that any two vertices are either not connected or connected by only one edge.

Let's add one more hypothesis to this design: given two treatments $\boldsymbol{i}$ and $\boldsymbol{j}$ there is a unique treatment $\boldsymbol{l}$ such that $\boldsymbol{i}$ and $\boldsymbol{l}$ occur in the same block ($\boldsymbol{i}$ e $\boldsymbol{l}$ are friends) and $\boldsymbol{j}$ e $\boldsymbol{l}$ also occur in the same block, that is, $\boldsymbol{l}$ is only common friend $\boldsymbol{i}$ and $\boldsymbol{j}$. Note that $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{l}$ may or may not occur in the same block and that all blocks are of size 3.

If there was a block with more than 3 treatments, for example 4 treatments, two treatments in that block would have two friends in common. Therefore, the concurrency graph of this design satisfies the conditions of the Friendship Theorem and is given by the graph in figure 11. A same treatment occurs in all blocks. This treatment that occurs in all blocks is usually called the control treatment. This design was called the “queen-bee design” (Bailey and Cameron, 2011).

Conclusions

The intensive use of graph geometry in the proof of the Friendship Theorem aims to make the proof more direct and understandable and, therefore, accessible to a non-specialist audience. The application of the theorem to an experimental design is a way of alerting statisticians to a current fact, the increasing use of graph theory as an effective tool in the theory and construction of experimental designs.

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- *RESUMO: Uma demonstração do Teorema da Amizade, resultado clássico em combinatória, é apresentada. Grafos são utilizados de forma intensiva com o objetivo de explicitar todos os passos da demonstração e assim torná-los mais intuitivos. Uma aplicação em delineamentos experimentais é apresentada.*
- *PALAVRAS-CHAVE: Grafos regulares; autovetores; delineamentos parcialmente balanceados.*

References

BAILEY, R. A.; CAMERON, P. J. Combinatorics of optimal desig*ns*. *Surveys in combinatorics*, v.365, n.19-73, p. 3, 2009.

BAILEY, R. A.; CAMERON, P. J. *Using graphs to find the best block designs,* 2011.

CALEGARI, R. S. *Demonstrações do Teorema da Amizade.* MS thesis. Brasil, 2018.

CASARIN JR, M. A.; TOMEI, C. *Uma demonstração elementar do teorema da amizade,* Matemática Universitária, v.6, 1987.

CHATTERJEE, D. *The Friendship Theorem,* 2014.

LIMA, E. L. *Álgebra linear*, 1.ed. Rio de Janeiro: IMPA, 2014.

VAN LINT, J. H.; WILSON, R. M.; WILSON, R. M. *A course in combinatorics*. Cambridge: Cambridge University Press, 2001.

WALKER, E. *The Friendship Theorem* (2016). Disponível em: http://math.mit.edu/apost/courses/18.204-2016/18.204ElizabethWalkerfinalpaper.pdf . (Acesso em 10 julho. 2016).

THEOREM OF THE DAY. Disponível em: https://www.theoremoftheday.org/CombinatorialTheory/Friendship/TotDFriendship.pdf. (Acesso em 22 julho. 2016).

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